THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fiftieth regular meeting of the Society was held in New York City on Saturday, October 29, 1910. The attendance at the morning and afternoon sessions included the following forty-three members:

Mr. F. W. Beal, Professor G. D. Birkhoff, Professor E. W. Brown, Mr. R. D. Carmichael, Dr. Emily M. Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Dr. Elizabeth B. Cowley, Professor E. S. Crawley, Professor L. P. Eisenhart, Professor H. B. Evans, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Professor C. C. Grove, Professor C. O. Gunther, Professor H. E. Hawkes, Dr. Frank Irwin, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. W. C. Krathwohl, Dr. N. J. Lennes, Professor J. H. Maclagan-Wedderburn, Dr. H. F. MacNeish, Dr. H. H. Mitchell, Professor Richard Morris, Professor G. D. Olds, Professor W. F. Osgood, Dr. H. W. Reddick, Mr. L. P. Siceloff, Dr. L. L. Silverman, Mr. C. G. Simpson, Professor D. E. Smith, Professor P. F. Smith, Mr. W. M. Smith, Professor Virgil Snyder, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White.

Ex-President W. F. Osgood occupied the chair at the morning session, Ex-President H. S. White and Professor Kasner at the afternoon session. The Council announced the election of the following persons to membership in the Society: Dr. G. A. Campbell, American Telephone and Telegraph Company; Mrs. E. B. Davis, Nautical Almanac Office; Professor C. W. Emmons, Simpson College; Professor H. C. Feemster, York College; Mr. R. R. Hitchcock, University of North Dakota; Mr. W. J. Montgomery, University of Michigan; Professor C. C. Morris, Ohio State University; Mr. H. S. Newcomer, University of Wisconsin; Professor A. D. Pitcher, University of Kansas; Professor George Rutledge, Georgia School of Technology. Four applications for membership in the Society were received.

The list of nominations for officers and other members of the

Council to be placed on the official ballot for the annual meeting was adopted. Provision was made for the appointment of a committee to audit the Treasurer's accounts. The Council decided to undertake the republication of the Evanston Colloquium Lectures and to place them on sale at a nominal price.

The following papers were read at the October meeting:

(1) Professor G. A. MILLER: "The group generated by two conjoints."

(2) Professor O. E. GLENN: "The conditions that the p-ary form of order m be a perfect mth power."

(3) Professor EDWARD KASNER: "A second converse of the theorem of Thomson and Tait."

(4) Dr. L. L. SILVERMAN: "Generalized definitions of the sum of divergent series."

(5) Dr. H. H. MITCHELL: "Note concerning a collineation group in n variables."

(6) Mr. R. D. CARMICHAEL: "Mixed equations and their analytic solutions" (preliminary communication).

(7) Professor G. A. MILLER: "Groups generated by two operators satisfying two conditions."

In Professor Miller's absence his papers were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. One of the most useful theorems proved by Jordan in his first published paper is that with every regular group H of degree n there is associated another regular group H' on the same letters such that every substitution of each of these two groups is commutative with every substitution of the other. The groups H and H' are known as conjoints, and the group G generated by H and H' has received considerable attention. In particular, Maillet observed that the necessary and sufficient condition that G be primitive is that H is a simple group. If H is abelian, H and H' are identical. This trivial case is excluded in what follows. Maillet considered also in his grand prix memoir of 1896 the class of G when H is a simple group of composite order and observed that an inferior limit is $\frac{4}{5}n$. This limit is also given in the Encyclopédie des Sciences mathématiques, tome 1, volume 1, page 593. Professor Miller proves that the inferior limit of the class of G when H is simple is really $\frac{11}{12}n$, and that the simple group of order 60 is the only one which gives such a low limit. When H is com-

posite, the inferior limit is $\frac{1}{2}n$. Moreover, G is always simply transitive and each of its substitutions which omits at least one letter must omit a number of letters which is divisible by its order. When G is primitive we thus have an infinite category of simply transitive primitive groups in which every substitution which omits one letter must omit at least two letters. Professor Cole called attention to a simply transitive primitive group of degree 9 which involves some substitutions with this property and observed that no such group exists whose degree is less than 9.* Professor Miller's paper has been offered for publication in *Prace Matematyczno-Fizyczne.*

2. By making use of certain results on factorable quantics which he has published previously, Professor Glenn proves that the necessary and sufficient conditions that a p-ary m-ic be a perfect mth power consist (1) in the

$$\binom{m+p-1}{p-1} - m(p-1) - 1$$

conditional relations which exist among the coefficients of the form in order that it be factorable into linear factors, and (2) the identical vanishing of $p-1$ binary Hessian determinants. The explicit forms of these Hessians, for p and m both general, are developed.

3. The purely geometric part of the theorem of Thomson and Tait suggested the converse question discussed by Professor Kasner in a former paper (*Transactions*, April, 1910). The present converse relates to the distribution of velocities along an initial base surface. In general the only possible distributions leading to normal congruences of trajectories are those for which the sum of the kinetic and potential energies is constant. The exceptional cases are limited and are discussed completely for central forces.

4. All of the definitions hitherto proposed for the sum of a divergent series have aimed to cover an increasingly wide range of series that will have a meaning. It then becomes desirable to extend theorems involving series to the case where the series diverge. In such cases, however, the extension of the theorem

*BULLETIN, vol. 2 (1893), p. 258.

in question to any new definition would require either a different proof, or at least a restatement of the proof given for previous definitions. To avoid the necessity for different proofs or for restatements of the same proof, Dr. Silverman states a definition for the sum of a series which not only contains all of the well-known definitions of summability, but includes all definitions of a certain type which are subject to the following requirements:

(i) The generalized sum must exist and be equal to the ordinary sum whenever the series converges.

(ii) If a term u_0 is dropped from a series with generalized sum s, the resulting series has a generalized sum, which is $s - u_0$.

(iii) If two series with s and t for generalized sums are added term by term, the resulting series has a generalized sum which is $s + t$.

A series $u_1 + u_2 + \cdots + u_n + \cdots$ is defined to be *evaluable** if there exists a set of functions $a_i(\alpha)$, where (α) is any assemblage with a cluster point at $+\infty$, such that the following conditions are fulfilled when $s_n = u_1 + u_2 + \cdots + u_n$:

1) $a_i(\alpha) \geq 0$, 2) $\lim_{\alpha=\infty} a_i(\alpha) = 0$ for every i,

3) $\lim_{n=\infty} \sum_{i=1}^{n} a_i(\alpha) \equiv 1$, 4) $\lim_{\alpha=\infty} \lim_{n=\infty} \sum_{i=1}^{n} a_i(\alpha) s_i = s$,

5) $\lim_{\alpha=\infty} \lim_{n=\infty} \sum_{i=1}^{n} a_i(\alpha) u_i = 0$.

This definition satisfies all of the requirements above. It includes, too, all of the definitions of summability which also satisfy these requirements. For $a_i(\alpha) = e^{-\alpha} \cdot \alpha^i / i!$, we obtain Borel's mean summability.† By putting

$$a_i(\alpha) \equiv a_i(n) \equiv \frac{r(r+1)\cdots(r+n-i)}{r-i+1} \Big/ \frac{(r+1)\cdots(r+n)}{n!},$$

we have Cesàro's summability of order r. Other definitions

* It is convenient to have a new name for series which have a value according to this general definition; a series will then be called *summable*, if it has a value for a particular set of $a_i(\alpha)$.

† Note: Borel's integral definition cannot be included since it fails to satisfy the second requirement. To meet this difficulty Borel introduces absolute summability; but this definition fails to satisfy the first requirement. See Bromwich, An Introduction to the Theory of Infinite Series, pp. 273 and 279.

may of course be given by specializing the $a_i(\alpha)$. Thus, e. g.,

$$a_i(\alpha) \equiv a_i(n) \equiv \frac{1}{i \log n}$$

gives

$$s = \lim_{n=\infty} \left(s_1 + \frac{s_2}{2} + \cdots + \frac{s_n}{n} \right) \Big/ \log n,$$

which has proved useful in connection with Dirichlet series. *

The paper will be published in the *Studies of the University of Missouri.*

5. In a paper concerning commutative collineations, in a recent number of the *Annals of Mathematics,* Professor W. B. Fite has called attention to an abelian group of order n^2 in n variables. This group may be generated by the two transformations

$$S: \quad x_i' = \omega^i x_i, \quad T: \quad x_i' = x_{i+1},$$

where i is to be taken modulo n and ω is an nth root of unity.

This group of order n^2 is invariant under a group of order

$$n^5 \left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{q^2}\right) \cdots,$$

(where $n = p^\alpha q^\beta \cdots$) which was discovered by Klein. If n is prime, this group is of order $(p+1)p^3(p-1)$, and it is proved by Dr. Mitchell to be simply isomorphic with the group of all transformations of the form

$$y' = a_1 y + a_2 z + a_3, \quad z' = b_1 y + b_2 z + b_3,$$

where the coefficients are marks of the $GF(p)$ and

$$a_1 b_2 - a_2 b_1 = 1.$$

6. In this preliminary communication Mr. Carmichael points out the existence in general of an infinite number of solutions of the mixed system of equations

$$D_x f_i(x) = \sum_{j=1}^{n} \phi_{ij}(x), f_j(x) \qquad (i = 1, \cdots, n),$$

where

$$\phi_{ij}(x) = a_{ij} x^{-2} + a'_{ij} x^{-3} + \cdots \qquad (|x| \geqq R),$$

* Ruiz: *Comptes Rendus,* July 5, 1909.

$D_i\,(i = 1, \cdots, k)$ denoting D the symbol of differentiation, and $D_i\,(i = k + 1, \cdots, n)$ denoting Δ the symbol of differences.

7. The first part of Professor Miller's second paper is devoted to a consideration of all the possible groups which can be generated by two operators s_1, s_2 when these operators satisfy two conditions of the form

$$s_1^{\alpha_1} s_2^{\alpha_2} s_1^{\alpha_3} \cdots = 1, \quad s_1^{\beta_1} s_2^{\beta_2} s_1^{\beta_3} \cdots = 1,$$

where each one of a set of consecutive exponents is unity while each of the other exponents is zero. He proved the following theorem: If identity can be obtained in two ways by forming the continued product with s_1, s_2 taken alternately as factors, then the largest group G generated by s_1, s_2 is of finite order except when the number of factors in both products is even, or when the number of these factors is the same odd number in both products and the same operator occurs an odd number of times in each. When the total number of times that each factor appears in the two products is the same and the number of factors in each product is odd then G is a cyclic group whose order is this total number. Among other theorems proved in this paper is the following: If two operators satisfy the condition $(s_1 s_2)^\alpha = (s_2 s_1)^\beta$, α and β being relatively prime, the order of $s_1 s_2$ may have an arbitrary value n, which is prime to α and β, and hence $s_1 s_2 = (s_2 s_1)^\gamma$. The order of s_2 is an arbitrary multiple of the exponent e to which γ belongs modulo n and hence the order of G is a multiple of en. All such groups are solvable.

F. N. COLE,
Secretary.

THE KÖNIGSBERG MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE twenty-first annual meeting of the Deutsche Mathematiker-Vereinigung was held at Königsberg, Prussia, September 18–24, 1910, in affiliation with the eighty-second convention of the Society of German naturalists and physicians.

A general reception for all the sections was held at the Festhalle of the Tiergarten on the evening of September 18; the division into sections and the organization of each took place the following morning. Six sessions of the Vereinigung

were held, under the chairmanship of Professors Hilbert, Meyer, v. Mangoldt, and Schoenflies.

The following papers were read:

(1) R. FUETER, Basel, "Fields of classes of complex multiplication, and its influence on the development of the theory of numbers."

(2) D. HILBERT, Göttingen, "On diophantine differential equations."

(3) A. HAAR, Göttingen, "Illustrations of and additions to the principles explained by Professor Hilbert."

(4) P. KOEBE, Göttingen, "Concerning conformal representation of multiply connected regions."

(5) E. MÜLLER, Vienna, "Some groups of theorems regarding oriented circles in the plane."

(6) F. MEYER, Königsberg, "A generalization of the concept of curvature."

(7) H. LIEBMANN, Leipzig, "On elementary constructions in non-euclidean space."

(8) L. BIEBERBACH, Göttingen, "On the group of motions in euclidean geometry."

(9) F. ENGEL, Greifswald, "A generalization of the infinitesimal parallel transformation."

(10) E. PAPPERITZ, Freiberg, "On drawing in space" (with the exhibition of apparatus for kinodiaphragmatic projection).

(11) A. SCHOENFLIES, Königsberg, "On the concept of definitions."

(12) A. WITTING, Dresden, "Communications concerning some manuscripts of Newton."

(13) W. v. IGNATOWSKI, Berlin, "The principle of relativity."

(14) F. MEYER, Königsberg, "On the theory of rotations."

(15) O. TOEPLITZ, Göttingen, "Some applications of the theory of an infinite number of variables."

(16) H. WEYL, Göttingen, "Developments in series, and the integral representation of arbitrary functions."

(17) A. SOMMERFELD, Munich, "Development of Green's function by means of characteristic functions."

(18) W. KREBS, Grossflotbeck, "New developments in the spectral photography of the sun."

(19) W. KREBS, Grossflotbeck, "Solar activity as volcanic action. Foundations of the theory."

In addition to the preceding papers, five others were read

before the joint meeting of the sections of mathematics, physics, and geodesy. They were devoted to an estimate of the influence of Bessel and of F. Neumann. The business meeting of the society occupied the entire morning session of September 21. Professor Schur, of Strassburg, was elected president for the coming year. Professors Dyck, of Munich, and Lilienthal, of Münster, were elected to the Council. The next meeting of the Vereinigung will be held at Carlsruhe in September, 1911.

Abstracts of some of the papers are given below, the numbers corresponding to those in the list above.

1. Kummer recognized that in the theory of algebraic numbers the field of the investigation must be extended until all the properties can be recognized. Apart from the concept of ideals, the Kronecker theory of complex multiplication forms the most important recent extension, since it brings in the aid of another hyperfield in order to develop the theory of numbers of an imaginary quadratic field. Besides the known concepts of field, rings, orders, moduli, it is also necessary to consider regions in which numbers of the set remain such by multiplication and division. Professor Fueter has studied such sets, which he here calls rays. Ideals are divided into classes of rays. If any equation in a field K is given, a definite ray belongs to it, connecting it with the known theory. Equations are made of exponential functions $e^{2\pi iz}$, z being rational. If we consider the law of factoring of the prime ideals of K in the hyperfield, we obtain the law of reciprocity. By this means we may calculate the number of classes in the field. For the case in which the ray is the field itself, Hilbert and Furtwängler have constructed the associated equations by arithmetic means.

5. Professor Müller emphasized the importance of cyclographical representation for geometric investigations and gave some illustrations. He obtained, for example, from the theorem: if of the four pairs of tangent planes which can be passed through the sides of a simple quadrilateral to a curve of class 2, four planes pass through a point, then the remaining ones also pass through a point, by means of cyclographic representation, a theorem regarding directed circles and straight lines, of which a number of theorems of Steiner (*Crelle*, volume 11 (1831), pages 117–129) are special cases. He showed further that the Feuerbach theorem, dualized in space, becomes:

if t_1, t_1'; t_2, t_2'; t_3, t_3' are pairs of points of a quadratic cone, the lines joining the points of a pair passing through a fixed point, then the planes $(t_1t_2t_3')$, $(t_1t_2't_3)$, $(t_1't_2t_3)$, $(t_1't_2't_3')$ cut the cone in four curves, touched by a plane section of the surfaces. In this way only can the theorems in the plane express their whole meaning.

9. If the expression $ds = \omega(x, y, y')dx$ is taken for element of arc, the equation $ds = \delta t$ defines an infinitesimal circle of radius δt and center x, y. If these circles are constructed for all points of a curve $y = f(x)$, there results, as envelope, an infinitesimally different curve, obtained from the formula by an infinitesimal contact transformation. A particular case is the infinitesimal parallel transformation of the euclidean plane. If now an arbitrary contact transformation is performed upon the element of arc, a new form is obtained,

$$ds = [\alpha(x, y, y') + \beta(x, y, y')y'']dx,$$

and the contact transformation previously defined is invariant with regard to contact transformations. A direct definition of this transformation is possible, based upon the new form of the element of arc. The purpose of Professor Engel's paper was to supply such a definition.

10. Professor Papperitz discussed the operations of drawing in space, presenting his apparatus for kinodiaphragmatic projection. The apparatus serves the purpose of representing geometric figures by means of projection of movable slits and in fact makes it possible to represent sharply and exactly the most diverse plane and space curves, and to illustrate changes of form produced by changing constants. The images in space are visible from every side. The process is patented; it should materially aid in the instruction in geometry. A prospectus will soon be published by Teubner. The fundamental idea is the following: The three necessary provisions for the representation are 1) adjustable source of light, 2) the original curve, 3) screen; the first is retained, but the others may be replaced by new apparatus, which will actually generate the image. These generators are constructed by mathematical rules, and select rays of light, part being reflected, part absorbed. Illustrations of systems of conics and of curves or quadrics were exhibited.

12. Dr. Witting reported on the preservation and the condition of the manuscripts of Newton in the library of Cambridge University, and gave some details of unpublished papers on mathematics, relating particularly to early investigations on the calculus.

At the joint meeting Professor Wangerin considered Franz Neumann (1798–1895) first in connection with mathematical physics, in which he was the first and for a long time the ablest German representative; he then discussed his achievements as an academic teacher, and finally as a mathematician.

A publication of 1826 on the Apollonius tactic problem is not well known; while the method is similar to that of French writers of the time, the treatment is more general in that contact is replaced by intersection at given angles, and both cones and spheres are discussed.

There are three purely arithmetic papers on the theory of harmonic functions. In the first (1828), the problem is solved of expanding an unknown function of the latitude and longitude based upon observations into a finite series of Laplace functions. The second (1848) discusses the application of harmonic functions to problems in the theory of the potential on surfaces of revolution, in particular to determine the distribution of terrestrial magnetism. In this paper the Neumann integral first appears. The third is an extensive essay (1878) concerning adjoint spherical harmonics in which the secondary index is larger than the principal index. A method is perfected by which the product of two spherical harmonics can be expanded in a series of harmonic functions. In many other investigations analytical skill of high order is noticeable, particularly in crystallography. Least squares were first applied to harmonize crystallographic measurements. Experimental researches in specific heat were based upon analytic developments (1830–1834). In optics new roads were broken, particularly in double refraction (1832), reflection of metals (1837), color of biaxial crystals in polarized light (1834), and many others, all based on mathematical foundations. Since 1840 most of Neumann's investigations were in connection with electricity.*

*See Wangerin: "Franz Neumann, sein Wirken als Forscher und Lehrer," Braunschweig, Vieweg, 1907.

ON THE SADDLEPOINT IN THE THEORY OF MAXIMA AND MINIMA AND IN THE CALCULUS OF VARIATIONS.

BY PROFESSOR R. G. D. RICHARDSON.

(Read before the American Mathematical Society, April 30, 1910.)

Introduction.

LAGRANGE has shown that the problem of determining a function $y(x)$ which satisfies the boundary conditions

$$y(0) = y(1) = 0 \tag{1}$$

and the integral condition

$$\int_0^1 g(x, y, y')dx = 0 \tag{2}$$

and which minimizes the integral

$$\int_0^1 f(x, y, y')dx \tag{3}$$

is equivalent, as far as the first variation is concerned, to the problem of minimizing the integral

$$\int_0^1 (f + \lambda g)dx, \tag{4}$$

the function being subject to no isoperimetric condition. The two constants of integration and the isoperimetric constant λ of the Lagrange differential equation

$$\frac{d}{dx}(f_{y'} + \lambda g_{y'}) + f_y + \lambda g_y = 0$$

which furnishes the solution are determined from the conditions (1) and (2). On the other hand it is possible to consider the problem of minimizing the integral (4) subject only to the boundary condition (1), in which case the minimum is obviously a function of the parameter λ. The determination of that value of λ which maximizes this minimum is a saddlepoint problem. The methods of this paper suffice to show that the first necessary condition for a solution $y(x)$ of this problem is identical with that for a solution of the foregoing isoperimetric problem.

Of a similar nature is the problem of minimizing the integral (3) for those values of $y(x)$ which satisfy the integral condition

$$\int_0^1 \{g(x, y, y') + \kappa h(x, y, y')\}dx = 0.$$

The minimum is a function $m(\kappa)$ of the parameter κ. The maximizing of $m(\kappa)$, that is, the determination of $y(x, \kappa)$ in such a way that it furnishes a maximum of a minimum is again a saddlepoint problem. It will be shown (§ 2) that the first necessary condition for a solution is identical with that for a solution of the problem of finding a minimum of (3) for those functions which satisfy the conditions

$$\int_0^1 g dx = 0, \quad \int_0^1 h dx = 0.$$

In the theory of maxima and minima there are also saddlepoint problems and related isoperimetric problems quite analogous to the foregoing and their treatment being naturally simpler is first considered in the discussion. In § 3 is given an application to an example which arises in the study of n self-adjoint linear differential equations of the second order containing n parameters.

The corresponding theory for the case of more general isoperimetric conditions is reserved for a later discussion.

§ 1. *An Extremum of an Extremum in the Theory of Maxima and Minima.*

Any solution of the problem of minimizing a function $f(x_1, \cdots, x_n)$ for those values of the variables which are subject to the relation $g(x_1, \cdots, x_n) = 0$ must satisfy the equations

$$\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0 \qquad (i = 1, \cdots, n), \tag{5}$$

$$g(x_1, \cdots, x_n) = 0. \tag{6}$$

In case a solution exists, the variables $\lambda, x_1, \cdots, x_n$ are determined from the $n + 1$ equations (5) and (6).

Related to the preceding problem is the following: The minimum of the function $f(x_1, \cdots, x_n) + \lambda g(x_1, \cdots, x_n)$ is a function $m(\lambda)$ of the parameter λ; it is required to maximize this minimum considered as a function of λ. In case a minimum

exists, the variables $x_1, \cdots, x_n$ are determined from the equations

$$\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0 \qquad (i = 1, \cdots, n). \tag{7}$$

To determine the maximum $m(\lambda)$ it is necessary to add to (7) the condition

$$\frac{\partial}{\partial \lambda}(f + \lambda g) \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \lambda} + \lambda \frac{\partial g}{\partial x_i} \frac{\partial x_i}{\partial \lambda} \right) + g = 0. \tag{8}$$

That the equations (7) and (8) are equivalent to (5) and (6) may be readily seen by multiplying the equations (7) by $\partial x_1/\partial \lambda, \cdots, \partial x_n/\partial \lambda$ respectively, and subtracting their sum from (8).

This result may be stated as follows: *The minimum of* $f(x_1, \cdots, x_n) + \lambda g(x_1, \cdots, x_n), (n > 1)$, *for all values of* $x_1, \cdots, x_n$ *is a function of the parameter* λ *which when maximized gives the same constant as the minimum of* f *for those values of the variables which satisfy the relation* $g = 0$.

The minimum of the function $f(x_1, \cdots, x_n)$ for those values of the variables which satisfy the relations $g(x_1, \cdots, x_n) = 0$, $h(x_1, \cdots, x_n) = 0$ is found by means of the equations

$$\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} + \mu \frac{\partial h}{\partial x_i} = 0 \qquad (i = 1, \cdots, n), \tag{9}$$

$$g(x_1, \cdots, x_n) = 0, \quad h(x_1, \cdots, x_n) = 0. \tag{10}$$

The minimum of the function $f(x_1, \cdots, x_n)$ for those values of the variables which satisfy the relation

$$g(x_1, \cdots, x_n) + \kappa h(x_1, \cdots, x_n) = 0$$

is a function of the parameter κ which when maximized must satisfy the equations

$$\frac{\partial f}{\partial x_i} + \lambda \left(\frac{\partial g}{\partial x_i} + \kappa \frac{\partial h}{\partial x_i} \right) = 0, \tag{11}$$

$$g(x_1, \cdots, x_n) + \kappa h(x_1, \cdots, x_n) = 0, \tag{12}$$

$$\frac{\partial f}{\partial \kappa} \equiv \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \kappa} = 0. \tag{13}$$

Proceeding now to show that the sets of equations (9), (10) and (11), (12), (13) are equivalent, we multiply the equations (11)

by $\partial x_1/\partial\kappa, \cdots, \partial x_n/\partial\kappa$ respectively, from the sum subtract (13), and obtain

$$\lambda\sum_{i=1}^{n}\left(\frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial\kappa}+\kappa\frac{\partial h}{\partial x_i}\frac{\partial x_i}{\partial\kappa}\right)\equiv\lambda\left(\frac{dg}{d\kappa}+\kappa\frac{dh}{d\kappa}\right)=0.$$

Disregarding the case where $\lambda=0$,* this becomes

$$\frac{dg}{d\kappa}+\kappa\frac{dh}{d\kappa}=0. \tag{14}$$

On the other hand by differentiating (12) with regard to κ, it follows that

$$\frac{dg}{d\kappa}+\kappa\frac{dh}{d\kappa}+h=0. \tag{15}$$

and from (14), (15) and (12) that

$$h(x_1, \cdots, x_n)=0, \quad g(x_1, \cdots, x_n)=0.$$

On setting $\lambda\kappa=\mu$ the complete equivalence of the two sets of equations is established and the first necessary conditions for the two problems are seen to be identical.

Since these methods of proof are perfectly general, the result may be enunciated as follows: *The minimum of the function* $f(x_1, \cdots, x_n)$ *for those values of the variables which satisfy the relation*

$$g(x_1, \cdots, x_n)+\sum_{i=1}^{m}\kappa_i h_i(x_1, \cdots, x_n)=0 \quad (m+1<n)$$

is a function of $\kappa_1, \cdots, \kappa_m$. *The first necessary condition for a maximum of this function is identical with that for a minimum of* $f(x_1, \cdots, x_n)$ *for those values of* $x_1, \cdots, x_n$ *which satisfy the relations* $g=0, h_1=0, \cdots, h_m=0$.

§ 2. *An Extremum of an Extremum in the Calculus of Variations.*

The problem of determining a function $y(x)$ which satisfies the boundary conditions

$$y(0)=y(1)=0 \tag{16}$$

and the integral conditions†

* In case $\lambda=0$, the solution is determined from the equations $\partial f/\partial x_i=0$ $(i=1, \cdots, n)$. The minimum is then independent of κ and the result we are seeking to establish is self-evident.

† On setting $g_1=g-c$, the case of the more general integral condition $\int_0^1 g\,dx=c$ is reduced to $\int_0^1 g_1\,dx=0$.

$$(17)\qquad \int_0^1 g(x, y, y')dx = 0, \qquad \int_0^1 h(x, y, y')dx = 0$$

and which minimizes the integral

$$(18)\qquad \int_0^1 f(x, y, y')dx$$

leads to the Lagrange equation

$$(19)\qquad \frac{d}{dx}(f_{y'} + \lambda g_{y'} + \mu h_{y'}) - (f_y + \lambda g_y + \mu h_y) = 0.$$

On the other hand the minimum of the integral (18) for those functions $y(x)$ which satisfy the boundary conditions (16) and the integral condition

$$(20)\qquad \int_0^1 \{g(x, y, y') + \kappa h(x, y, y')\}dx = 0$$

is furnished by the equation

$$\frac{d}{dx}\{f_{y'} + \lambda(g_{y'} + \kappa h_{y'})\} - \{f_y + \lambda(g_y + \kappa h_y)\} = 0.$$

The condition that this minimum be maximized as a function of the parameter κ is identical with that of finding a maximum of the function

$$\phi(\epsilon_1) = \int_0^1 f(x, \bar{y} + \epsilon_1\eta, \bar{y}' + \epsilon_1\eta')dx$$

subject to the isoperimetric condition

$$\psi(\epsilon_1, \epsilon_2) = \int_0^1 \{g(x, \bar{y} + \epsilon_1\eta, \bar{y}' + \epsilon_1\eta') + (\bar{\kappa} + \epsilon_2)h(x, \bar{y} + \epsilon_1\eta, \bar{y}' + \epsilon_1\eta')\}dx = 0,$$

where $\bar{y}(x)$ and $\bar{\kappa}$ are the values of the function and parameter which furnish the extremum of the extremum, η an admissible variation arbitrarily chosen, and ϵ_1, ϵ_2 infinitesimals. On differentiating $\phi + \lambda\psi$ with regard to ϵ_1, ϵ_2 respectively and equating the results to zero for $\epsilon_1 = \epsilon_2 = 0$, one obtains the necessary conditions

$$(21)\qquad \int_0^1 [\{f_{y'} + \lambda(g_{y'} + \bar{\kappa}h_{y'})\}\eta' + \{f_y + \lambda(g_y + \bar{\kappa}h_y)\}\eta]_{y=\bar{y}}dx = 0,$$

$$(22)\qquad \int_0^1 h(x, \bar{y}, \bar{y}')dx = 0.$$

Since η is an arbitrary function vanishing at the points $x = 0$, $x = 1$, by applying the product integration usual in such cases, it is readily shown that on setting $\mu = \bar{\kappa}\lambda$, equation (21) is equivalent to (19). Since equations (20) and (22) are equivalent to (17), it follows that the first necessary condition which a solution of this saddlepoint problem must satisfy is identical with that in the ordinary isoperimetric problem with two integral conditions.

The method here employed may be applied to the other problem proposed in the introduction; it admits also of immediate generalization to functions of several variables subject to integral conditions involving several parameters.

§ 3. *Application to an Example.*

As an application of the theory of the preceding section let us consider the problem * of finding functions $u_1(x), u_2(x)$ which satisfy the boundary conditions

$$u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0 \tag{23}$$

and the integral condition

$$\int_0^1 [l_1(x)u_1^2(x) + l_2(x)u_2^2(x) - 1 + \kappa\{r_1(x)u_1^2(x) + r_2(x)u_2^2(x)\}]\,dx = 0, \tag{24}$$

and which minimize the integral

$$D(u_1, u_2) = \int_0^1 \{p_1(x)u_1'^2(x) + p_2(x)u_2'^2(x) - q_1(x)u_1^2(x) - q_2(x)u_2^2(x)\}\,dx,$$

where $p_i > 0$, $q_i \leqq 0$, l_i, r_i, $(i = 1, 2)$, are analytic functions of x in the interval $(0, 1)$. In order that a minimum † exist for

* Other aspects of this problem are considered by the author in a paper to be published in the *Mathematische Annalen*.

† That a minimum exists for all values of κ and is equal to the smallest (say λ_1) of the positive characteristic numbers (Eigenwerte) $\lambda_1, \lambda_2, \cdots$; $\lambda_1', \lambda_2', \cdots$ of the Lagrange equations

$$(p_i u_i')' + q_i u_i + \lambda(l_i + \kappa r_i)u_i = 0 \quad (i = 1, 2) \tag{A}$$

may be proved as follows: Consider the identities

$$p_i u_i'^2 - q_i u_i^2 - \lambda(l_i + \kappa r_i)u_i^2 \equiv \left(\frac{p_i y_i' u_i^2}{y_i}\right)' + p_i y_i^2 \left(\frac{u_i}{y_i}\right)'^2 - \frac{u_i^2}{y_i}[(p_i y_i')' + q_i y_i + \lambda(l_i + \kappa r_i)y_i] \quad (i = 1, 2),$$

various values of the parameter κ it is necessary to exclude the case that both of the functions l_1, l_2 are positive throughout the whole interval and also the case that both the functions r_1, r_2 are positive or both negative throughout the interval. (For convenience of notation in the following discussion it is assumed that r_1 takes both signs in the interval.)

If then it is possible to show that the minimum $m(\kappa)$ of $D(u_1, u_2)$ considered as a function of κ possesses a maximum, it follows from the preceding theory that the functions $u_1(x)$, $u_2(x)$ which furnish this maximum of a minimum, furnish also a minimum for the integral $D(u_1, u_2)$ subject to the boundary conditions (23) and the integral conditions

$$\int_0^1 \{l_1(x)u_1^2 + l_2(x)u_2^2\}dx = 1, \quad \int_0^1 \{r_1(x)u_1^2 + r_2(x)u_2^2\}dx = 0.$$

In order to show that this maximum exists, we note that $D(u_1, u_2)$ can be zero only if $u_1 \equiv 0$, $u_2 \equiv 0$. Since these values will not satisfy (24), a constant $c > 0$ may be chosen such that for values of k within a restricted interval $m(k) > c$. Since m is a continuous analytic function of κ it follows that a maximum exists if it can be shown that $m(\infty) = 0$, $m(-\infty) = 0$. With this in view let us consider the function pair $u_1(x) \equiv u_1(x)$, $u_2(x) \equiv 0$. The condition (24) may then be written

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where $u_1(x)$, $u_2(x)$ are any two functions which vanish at the end points $x = 0$, $x = 1$. If $\lambda_0 < \lambda_1$, we must show that $D(u_1, u_2) > \lambda_0$ for all functions $u_1(x)$, $u_2(x)$ which satisfy (24). The functions $y_1(x)$, $y_2(x)$ are chosen to be solutions of the differential equations

$$(p_i y_i')' + q_i y_i + \lambda_0(l_i + \kappa r_i)y_i = 0 \quad (i = 1, 2)$$

which vanish neither within the interval nor at the end points. (This is possible since λ_0 is smaller than either of the characteristic numbers λ_1, λ_1' corresponding to those solutions of the differential equations (A) which vanish at the end points. See proof by author, *Mathematische Annalen*, vol. 68, No. 2, 1910.) On setting $\lambda = \lambda_0$ in the identities, integrating and adding, we have

$$\int_0^1 \Sigma(p_i u_i'^2 - q_i u_i^2)dx - \lambda_0 \int \Sigma(l_i + \kappa r_i)u_i^2 dx = \int_0^1 \Sigma p_i y_i^2 \left(\frac{u_i'}{y_i}\right)'^2 dx.$$

Since $p_1(x)$ and $p_2(x)$ are positive, this may be written $D(u_1, u_2) - \lambda_0 > 0$. That $D(u_1, u_2)$ takes the value λ_1 may be seen by setting $\lambda = \lambda_1$ in the first identity and integrating. For $u_2(x) \equiv 0$ this gives $D(u_1, u_2) - \lambda_1 = 0$ (since y_1 can be chosen equal to u_1). It may be noted that in case the characteristic numbers λ_1 and λ_1' are not equal, in order to furnish a maximum, one of the functions $u_1(x)$, $u_2(x)$ must be identically zero. In case however that $\lambda_1 = \lambda_1'$ the maximum can be furnished by the function pairs (1) $u_1(x) \neq 0$, $u_2(x) \equiv 0$, (2) $u_1(x) \equiv 0$, $u_2(x) \neq 0$, or (3) $u_1(x) \neq 0$, $u_2(x) \neq 0$.

$$\int_0^1 \left\{ \frac{l_1(x)}{\kappa} + r_1(x) \right\} u_1^2(x)dx = \frac{1}{\kappa}. \tag{25}$$

By taking κ large enough it is possible to choose $u_1(x)$ (in an infinite number of ways) such that equation (25) is satisfied and $|u_1(x)| < \epsilon$, $|u_1'(x)| < \epsilon$, where ϵ is an arbitrarily small positive number. For this value of $u_1(x)$ (and $u_2(x) \equiv 0$), $D(u_1, u_2) < C\epsilon^2$, where C is a constant such that $C > p_1(x) - q_1(x)$. The value of κ is then so chosen that $m(\kappa)$ is less then $C\epsilon^2$. In a similar manner it may be shown that $m(-\infty) = 0$.

BROWN UNIVERSITY, PROVIDENCE,
September, 1910.

NOTE ON IDENTITIES CONNECTING CERTAIN INTEGRALS.

BY DR. LOUIS INGOLD.

(Read before the American Mathematical Society, September 7, 1910.)

BECAUSE of the general nature of the symbols and symbolic products used in the symbolic invariant theory, it is possible to apply the formulas of this theory in various special fields. In the present note the theory is employed to obtain relations connecting integrals of functions constructed out of a linearly independent set.

1. We shall be interested in functions of n parameters $u_1, u_2, \cdots, u_n$ and functions of one or more real variables $x, x_1, x_2, \cdots, y, y_1, y_2, \cdots$, etc., restricted to a definite interval, say, $0 \leq x \leq 1$. In order to distinguish readily between those quantities which are constant or functions of the parameters $u_1, \cdots, u_n$ only, and those which are functions of one or more of the variables $x, x_1, x_2, \cdots$, etc., the latter will always be denoted by black faced letters; the others by letters in ordinary type; thus $\mathbf{a}$ denotes a function of a variable x defined for values of x on the interval $0 \cdots 1$, while a denotes a function of the u_i only or a constant. The partial derivatives $\partial \mathbf{a}/\partial u_i$, $\partial a/\partial u_i$, etc., will be denoted by $\mathbf{a}_{(i)}$, $a_{(i)}$, etc.

2. Consider now the total differential of any function $\mathbf{f}(x; u_1, u_2, \cdots, u_n)$ with respect to the u_i

$$d\mathbf{f} = \mathbf{f}_{(1)}du_1 + \mathbf{f}_{(2)}du_2 + \cdots + \mathbf{f}_{(n)}du_n. \tag{1}$$

Let this expression be squared and integrated with respect to x from 0 to 1. We obtain in this way the quadratic differential form in the variables $u_1, \cdots, u_n$

$$\int_0^1 (df)^2 dx = \int_0^1 (\mathfrak{f}_{(1)}du_1 + \mathfrak{f}_{(2)}du_2 + \cdots + \mathfrak{f}_{(n)}du_n)^2 dx$$

$$= \sum_{i,j=1}^{n} \left(\int_0^1 \mathfrak{f}_{(i)}\mathfrak{f}_{(j)}dx \right) du_i du_j = \sum_{i,j=1}^{n} E_{ij}du_i du_j, \tag{2}$$

where the expressions

$$E_{ij} = \int_0^1 \mathfrak{f}_{(i)}\mathfrak{f}_{(j)}dx \tag{3}$$

are functions of $u_1, u_2, \cdots, u_n$.

3. If now we introduce a symbolic multiplication equivalent to ordinary multiplication followed by integration with respect to x from 0 to 1, we may write

$$\overline{df^2} = (\overline{\mathfrak{f}_{(1)}du_1 + \mathfrak{f}_{(2)}du_2 + \cdots + \mathfrak{f}_{(n)}du_n})^2$$

$$= \sum_{i,j=1}^{n} \overline{\mathfrak{f}_{(i)}\mathfrak{f}_{(j)}}du_i du_j = \sum_{i,j=1}^{n} E_{ij}du_i du_j, \tag{4}$$

where the bars indicate that the products are symbolic, i. e., integrated products. We have thus expressed the quadratic differential form

$$E \equiv \sum_{i,j=1}^{n} E_{ij}du_i du_j$$

as the symbolic square of a linear form.*

4. It should be noticed that we have also

$$\int_0^1 [df(y)]^2 dy = \int_0^1 [df(x)]^2 dx = E \tag{5}$$

and again, if $\phi(x; u_1, u_2, \cdots, u_n)$ is any function satisfying the conditions

$$\int_0^1 \phi_{(i)}\phi_{(j)}dx = E_{ij}, \tag{3}$$

then

$$\int_0^1 [(d\phi(x)]^2 dx = E. \tag{6}$$

*Cf. Maschke, "A symbolic treatment of the theory of invariants of quadratic differential quantics of n variables," *Transactions*, vol. 4 (1904), pp. 445–469.

Such equivalent expressions for E may be employed in a manner analogous to the use of equivalent symbols in the symbolic theory.* In this paper the method indicated in equations (5) will be used for equivalent representation of E.

5. On account of the linear distributive character of integration, the ordinary function $\mathbf{f}(x\,;\ u_1, u_2, \cdots, u_n)$ satisfying conditions (3) may be taken as a *symbolic function* or *symbol* of the differential form E, and all the operations of the symbolic theory can be carried out if we agree that in products no more than two factors shall be expressed in terms of the same variable x; that products of two factors written in terms of the same variable x shall be regarded as *integrated* products; and that a product of any number of factors expressed in terms of different variables $x, x_1, x_2, \cdots, y, y_1, y_2, \cdots$ shall be regarded as ordinary products.

We have then this result:

All of the identities of the symbolic differential invariant theory† may be interpreted as identities involving ordinary functions and their integrals.

6. The identities of the symbolic theory are given mainly in terms of determinants of the nth order. We denote the determinant

$$\left|\int_0^1 \mathbf{f}_{(i)}\mathbf{f}_{(j)}dx\right| \quad \text{by} \quad \frac{1}{\beta^2}.$$

The product of β and the Jacobian of n functions $a_1, a_2, \cdots, a_n$, is denoted by

$$(a_1, a_2, \cdots, a_n).$$

Using these notations we may write, for example, the symbolic expression

$$(f_1, f_2, \cdots, f_k,\ a_1, a_2, \cdots, a_{n-k})(f_1, f_2, \cdots, f_k,\ b_1, b_2, \cdots, b_{n-k})$$

(where $f_1, f_2, \cdots, f_k$ are equivalent symbolic functions of the differential quantic E) in the form

$$\int_0^{1(k)} [\mathbf{f}(x_1), \mathbf{f}(x_2), \cdots, \mathbf{f}(x_k), a_1, \cdots, a_{n-k}] \times [\mathbf{f}(x_1), \cdots, \mathbf{f}(x_k), b_1, \cdots, b_{n-k}]dx_1, \cdots, dx_k,$$

where $\int_0^{1(k)}$ stands for $\int_0^1\int_0^1\cdots$ k times.

* Maschke, loc. cit., p. 448.

† Clearly, the result holds equally well for the algebraic invariant theory.

7. Out of the large number of identities which could now be written down, we select the following for illustration.

$$\begin{aligned}(7)\qquad &\int_0^{1^{(n-k)}}[\mathbf{f}(x_1), \cdots, \mathbf{f}(x_{n-k}), a_1, \cdots, a_k]\\ &\quad\times [\mathbf{f}(x_1), \cdots, \mathbf{f}(x_{n-k}), b_1, \cdots, b_k]dx_1, \cdots, dx_{n-k}\\ &= \frac{(n-k)!}{(n-1)!}\left|\int_0^{1^{(n-1)}}[\mathbf{f}(x_1), \cdots, \mathbf{f}(x_{n-1}), a_i]\right.\\ &\quad\left.\times [\mathbf{f}(x_1), \cdots, \mathbf{f}(x_{n-1}), b_j]dx_1, \cdots, dx_{n-1}\right|,*\end{aligned}$$

where the determinant on the right is of the kth order, i and j running from 1 to k.

$$\begin{aligned}(8)\qquad &\int_0^{1^{(k)}}[\mathbf{f}(x_1), \cdots, \mathbf{f}(x_k), \mathbf{f}(x_{k+1}), \cdots, \mathbf{f}(x_n)]\\ &\quad\times [\mathbf{f}(x_1), \cdots, \mathbf{f}(x_k), a_1, \cdots, a_{n-k}]\\ &\quad\times [\mathbf{f}(x_{k+1}), \cdots, \mathbf{f}(x_n), b_1, \cdots, b_k]dx_1, \cdots, dx_n\\ &= k!\,(n-k)!\,(a_1, \cdots, a_{n-k}, b_1, \cdots, b_k).\dagger\end{aligned}$$

$$\begin{aligned}(9)\qquad &\int_0^{1^{(n-k-r)}}[\mathbf{f}(x_1), \cdots, \mathbf{f}(x_k), \mathbf{f}(x_{k+1}), \cdots, \mathbf{f}(x_{n-r}), b_1, \cdots, b_r]\\ &\quad\times [\mathbf{f}(x_{k+1}), \cdots, \mathbf{f}(x_{n-r}), a_1, \cdots, a_{k+r}]dx_{k+1}, \cdots, dx_{n-r}\\ &= \frac{(n-k-r)!}{(n-k)!(n-r)!}\sum_i\int_0^{1^{(n-r)}}[\mathbf{f}(x_1), \cdots, \mathbf{f}(x_{n-r}), a_i, \cdots, a_{i_r}]\\ &\quad\times [\mathbf{f}(x_1), \cdots, \mathbf{f}(x_{n-r}), b_1, \cdots, b_r]dx_{k+1}, \cdots, dx_{n-r}\\ &\quad\times \int_0^{1^{(n-r)}}[\mathbf{f}(y_1), \cdots, \mathbf{f}(y_{n-k}), a_{i_{1+r}}, \cdots, a_{i_{k+r}}]\\ &\quad\times [\mathbf{f}(y_1), \cdots, \mathbf{f}(y_{n-k})\mathbf{f}(x_1), \cdots, \mathbf{f}(x_k)]dy_1, \cdots, dy_{n-k},\ddagger\end{aligned}$$

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* Maschke, "Differential parameters of the first order," *Transactions*, vol. 7 (1906), p. 74, formula (8).

† Cf. Maschke, loc. cit., p. 453, formula (34) and p. 455, formula (61). Only the cases $k = 1$ and $k = 2$ are given by Maschke but this extension is easily proved.

‡ This formula is given in my paper "Vector interpretation of symbolic differential parameters," *Transactions*, vol. 11, p. 470.

the summation extending over all distinct combinations $i_1 \cdots, i_r$ of r of the subscripts $1, 2, \cdots, k+r$ and the order $i_1, i_2, \cdots, i_{k+r}$ being so chosen that it may be obtained from $1, 2, \cdots, k+r$ by an even number of transpositions.

8. We now consider a special case in which $\mathbf{f}$ is linear in the u_i and a set of n normed and mutually orthogonal functions $\mathbf{f}_{(1)}, \mathbf{f}_{(2)}, \cdots, \mathbf{f}_{(n)}$, of the variable x

$$\mathbf{f} = u_1\mathbf{f}_{(1)} + u_2\mathbf{f}_{(2)} + \cdots + u_n\mathbf{f}_{(n)}. \tag{10}$$

We shall also suppose the functions a_i, b_i, etc., of the u's to be linear, and write

$$a_i = \sum_{j=1}^{n} \alpha_{ij}u_j, \quad b_i = \sum_{j=1}^{n} \beta_{ij}u_j, \quad \text{etc.}, \tag{11}$$

where the coefficients α_{ij}, β_{ij}, etc., are constant.

With every linear function a_i of the u's we shall associate a function $\mathbf{a}_i(x)$, defined by

$$\mathbf{a}_i(x) = \sum_{j=1}^{n} \alpha_{ij}\mathbf{f}_{(j)}. \tag{12}$$

We then introduce the notations

$$\mathbf{A}_{(x_1, \cdots, x_k)} = \frac{1}{k!}\begin{vmatrix} \mathbf{a}_1(x_1), & \cdots, & \mathbf{a}_1(x_k) \\ \mathbf{a}_2(x_1), & \cdots, & \mathbf{a}_2(x_k) \\ \mathbf{a}_k(x_1), & \cdots, & \mathbf{a}_k(x_k) \end{vmatrix} \tag{13}$$

$$\mathbf{F}(x_1, \cdots, x_n) = [\mathbf{f}(x_1), \mathbf{f}(x_2), \cdots, \mathbf{f}(x_n)]. \tag{14}$$

We define the *complement* of $\mathbf{A}$ (written $\bar{\mathbf{A}}$) by the formula

$$(\mathbf{A})_{(x_1, \cdots, x_{n-k})} = \int_0^{1(k)} \mathbf{F}(y_1, \cdots, y_k, x_1, \cdots, x_{n-k})\mathbf{A}_{(y_1, \cdots, y_k)}dy_1, \cdots, dy_k. \tag{15}$$

The following formula may then be easily proved :

$$k!(n-k)!\,\mathbf{A}_{(x_1, \cdots, x_k)} = \int_0^{1(n-k)} [\mathbf{f}(y_1), \cdots, \mathbf{f}(y_{n-k}), \mathbf{f}(x_1), \cdots, \mathbf{f}(x_k)] \times [\mathbf{f}(y_1), \cdots \mathbf{f}(y_{n-k}), a_1, \cdots, a_k]dy_1, \cdots, dy_{n-k}, \tag{16}$$

$$\mathbf{A} = \frac{1}{k!\,[(n-k)!]^3} \int_0^{1(k)} \mathbf{F}(y_1, \cdots, y_k x_1, \cdots, x_{n-k}) \tag{17}$$

$$\times \int_0^{1^{(n-k)}} \mathbf{F}(z_1, \cdots, z_{n-k} y_1, \cdots, y_k)$$

$$\times [\mathbf{f}(z_1), \cdots, \mathbf{f}(z_{n-k}), a_1, \cdots, a_k] dz_1, \cdots, dz_{n-k} dy_1, \cdots, dy_k$$

$$= \pm \frac{1}{(n-k)!} [\mathbf{f}(x_1), \cdots, \mathbf{f}(x_{n-k}), a_1, \cdots, a_k].$$

$$\bar{\bar{\mathbf{A}}} = \pm \mathbf{A}. \tag{18}$$

$$\int_0^{1^{(k)}} \mathbf{A}_{(x_1, \ldots, x_k)} \mathbf{B}_{(x_1, \ldots, x_k)} dx_1, \cdots, dx_k$$

$$\times \int_0^{1^{(n-k)}} (\bar{\mathbf{A}})_{(x_1, \ldots, x_{n-k})} (\bar{\mathbf{B}})_{(x_1, \ldots, x_{n-k})} dx_1, \cdots, dx_{n-k}, \tag{19}$$

Substituting in formulas (7), (8), and (9), we have

$$\int_0^{1^{(k)}} \mathbf{A}_{(x_1, \ldots, x_k)} \mathbf{B}_{(x_1, \ldots, x_k)} dx_1, \cdots, dx_k = \left| \int_0^1 \mathbf{a}_i \mathbf{b}_j dx \right|, * \tag{7'}$$

$$\int_0^{1^{(n-k)}} \mathbf{A}_{(x_1, \ldots, x_{n-k})} (\bar{\mathbf{B}})_{(x_1, \ldots, x_{n-k})} dx_1, \cdots, dx_{n-k}$$

$$= (-1)^{k(n-k)} \int_0^{1^{(k)}} (\bar{\mathbf{A}})_{(x_1, \ldots, x_k)} \mathbf{B}_{(x_1, \ldots, x_k)} dx_1, \cdots, dx_k, \tag{8'}$$

$$\int_0^{1^{(j)}} (\bar{\mathbf{A}})_{(y_1, \ldots, y_j)} (\bar{\mathbf{B}})_{(x_1, \ldots, x_k y_1, \ldots, y_j)} dy_1, \cdots, dy_j$$

$$= \frac{k!\, j!}{(k+j)!} \sum_i \mathbf{A}_{i(x_1, \ldots, x_k)} \int_0^{1^{(j)}} (\bar{\mathbf{A}}_i)_{(y_1, \ldots, y_j)} (\bar{\mathbf{B}})_{(y_1, \ldots, y_j)} dy_1, \cdots, dy_j \tag{9'}$$

where $k + j = n - r$ and where i has the same meaning as in formula (9).

UNIVERSITY OF MISSOURI,
July, 1910.

*This identity is proved directly by Professor R. G. D. Richardson and Dr. W. A. Hurwitz in the paper "Note on determinants whose terms are certain integrals," BULLETIN, vol. 16, p. 16.

POINCARÉ'S GÖTTINGEN LECTURES.

Sechs Vorträge über ausgewählte Gegenstände aus der reinen Mathematik und mathematischen Physik. Von HENRI POINCARÉ. Leipzig and Berlin, Teubner, 1910. 8vo. 60 pp.

IN these six Göttingen lectures, delivered April 22–28, 1909, at the invitation of the Wolfskehl commission, Poincaré has treated a wide range of interesting subjects in a masterly and illuminating way. It is in the nature of the case that the methods and results should be outlined with exceeding brevity, and this makes the little book hard reading. Fortunately, the reader who is not content with the summary discussions given in it can supplement them for the most part by the use of recent articles by Poincaré.* The topics in their order are (1) the Fredholm equations, (2) the application of the theory of integral equations to fluid motion, (3) the application of the theory of integral equations to Hertzian waves, (4) the reduction of Abelian integrals and the theory of Fuchsian functions, (5) transfinite numbers, (6) the new mechanics. The sixth lecture was popular in its nature and was delivered in the French language.

1. The Fredholm equations. The integral equation of the second kind

$$\phi(x) = \lambda \int_a^b f(x, y)\phi(y)dy + \psi(x)$$

is known to admit two formal solutions, namely the Neumann solution as a power series in positive integral powers of the parameter λ, which converges for small values of λ, and the Fredholm solution as the quotient of two entire functions of λ. Poincaré first derives the fundamental formula for $\log D(\lambda)$, where $D(\lambda)$ is the denominator of the Fredholm resolvent, by a count of combinations, and then defines the numerator at once by a use of the Neumann formula for the resolvent. By this method of comparison a clear analysis of the solution of the integral equation is obtainable. A natural extension of the method enables Poincaré to treat the important case where the

*First lecture: *Acta Mathematica*, vol. 33 (1909), pp. 57–86. Third lecture: *Palermo Rendiconti*, vol. 30 (1910), pp. 169–259. Fourth lecture: *Palermo Rendiconti*, vol. 29 (1909), pp. 281–336. Fifth lecture: *Acta Mathematica*, vol. 32 (1908), pp. 195–200, and *Revue de Metaphysique et de Morale*, 1909, pp. 461–482.

kernel $f(x, y)$ becomes infinite but some iterated kernel $f_n(x, y)$ remains finite. Fredholm has shown that a solution exists; but in his formulas there remained a common factor in numerator and denominator, which is here removed by the use of a modified resolvent. This resolvent is obtained in a very simple way by striking out certain terms from the Fredholm resolvent.

Some partial results for the case when $f(x, y)$ and all of its iterated kernels become infinite are stated, and the lecture concludes with a consideration of two special integral equations of the first kind, reducible to Fredholm equations by means of Fourier's integral and series.

2. Application of the theory of integral equations to fluid motion. In this lecture and the succeeding one are considered typical and important examples of the way in which problems of mathematical physics lead to Fredholm equations.

The first problem is to determine the fluid motion in a sea of varying depth on a rotating earth under the influences of periodic perturbative forces. If the attractive forces due to water displacement be neglected, there results a non-homogeneous linear partial differential equation of the second order in two independent variables. It is this equation which is considered. When a vertical wall surrounds the sea the method of Hilbert and Picard is applied to construct the kernel or Green's function of the associated integral equation, which is of the first kind. The equation can then be treated by a method due to Kellogg, or by a method of integration in the complex plane given by Poincaré, which replaces the given integral equation by an equivalent one of the second kind. If the shore of the sea is not vertical, the corresponding line is a singular line of the differential equation. Also a second line (critical latitude) is now taken into account. A similar reduction can be again effected by a succession of three steps. In this way a solution is proved to exist in both cases. Finally it is indicated that no new difficulty appears if one does not neglect the attraction due to water displacement.

3. Application of the theory of integral equations to Hertzian waves. Poincaré investigates the phenomenon of the curvilinear propagation of Hertzian waves on the earth's surface, which accounts for the great distances over which wireless telegraph messages may be sent. This phenomenon has been attributed to the great length of the Hertzian wave as compared with that of the light wave, which is propagated in a straight line. A quanti-

tative mathematical discussion is here given. By regarding the earth as the outer conductor and the transmitter as the inner conductor, and by considering damped synchronous vibrations, one obtains an integral equation of the second kind to determine the electrical density μ produced on the earth's surface. But this merely yields an existence theorem. The lecturer brings the problem to a practical conclusion by obtaining an approximate expression for μ; the method depends on an expansion of μ in Legendre polynomials, and the use of asymptotic formulas. It turns out that the curvature increases with an increase in the length of the wave, and with a decrease in the distance of the transmitter from the earth.*

4. Reduction of Abelian integrals and the theory of Fuchsian functions. If a system S of Abelian functions is given which is reducible in terms of a second system S', and if both S and S' arise from algebraic curves C and C', there will be an algebraic correspondence set up between groups of points on the curves; only that case is considered in which to one point of C corresponds one point of C', while to one point of C' correspond n points of C. There then exist certain reducible integrals which belong to C, and the table of periods has a simple normal form. Also the number n is equal to the order of a related theta function, as is proved. Now it is known that there exists a Fuchsian function which uniformizes any algebraic curve, in particular C'. The fundamental polygons for the curve C' may be taken to be limited by arcs of circles, and each such polygon for C to be formed by n of these polygons for C'. The interplay between the integrals and the polygons gives rise to numerous geometric facts about the curves C and C', and about congruent polygons in space of constant negative curvature. Several examples are given which illustrate the beautiful theorems concerning C and C' that may be thus obtained.

5. Transfinite numbers. The lecture develops Poincaré's attitude toward some of the subtleties in this controversial field of mathematics. The main ideas for which he contends are two in number: first that no mathematical entity exists that is not definable in a finite number of words, and secondly that all definitions must be what he calls 'predicative.' For example, Poincaré objects to the familiar proof that every algebraic equation $f(x) = 0$ has a root, dependent on the exis-

* It is noted at the end of the lecture that the final conclusions will need modification owing to the fact that important terms have been overlooked.

tence of a minimum of $|f(x)|$. For it would not be possible from his standpoint to speak of the totality of values of $f(x)$ without meaning to refer to those values for which x is defined in a finite number of words, and this is not permissible since the notion of the totality of definable values of x is non-predicative. The difficulty involved is that there are elements of this 'totality' which themselves are defined in terms of the 'totality,' and hence the notion embodies a vicious circle. The elucidation of the meaning of the word 'predicative' given in the lecture is not clear.

Poincaré begins by considering the apparent contradiction between Richard's proof (based on the first of the above-stated ideas) that the continuum is denumerable, and Cantor's proof that it is not denumerable, but it is shown that this contradiction is not real, since Richard employs a non-predicative definition.

He then passes on to point out how the demonstration of the theorem that every algebraic equation $f(x)=0$ has a root needs to be stated from his point of view.

Several other matters are briefly touched upon in conclusion. The Bernstein theorem is correct for Poincaré, and the problem of the well-ordering of the continuum (in Cantor's sense) seems meaningless to him. Moreover, he is not convinced that the second transfinite cardinal number exists. These conclusions are all in thorough-going harmony with the two main ideas.

The principal objection that may be urged against these views of Poincaré's (and of many other mathematicians) is the practical one that they so closely restrict the notion of a *class*. However it is the intuition rather than the logical faculty that rebels against their acceptance. Judging by the past, this fact is an indication in favor of the views presented; they form one step toward the elimination of the infinite from mathematics, and one may well doubt whether the conception of an infinite class as having objective existence will play any ultimate rôle in rigorous mathematics.

6. The new mechanics. In this closing popular lecture, the modifications that mechanics may have to undergo as a result of recent advances in physics are considered. If the final equations of motion are those of the electromagnetic field, and experiments seem to indicate this, startling conclusions follow: the velocity of all bodies is less than that of light and their inertia increases with their velocity; it will be

impossible to tell by any experiment whether one is at rest or in uniform motion of translation with respect to the ether; one cannot say that two events are simultaneous in an absolute sense; moreover, all bodies will undergo a shortening in the direction of their motion. It is with this fascinating subject, more especially with the modified mechanics that it implies, that Poincaré deals.

It is only in bodies that possess a very great velocity that one can hope to discern a deviation from the laws of the Newtonian mechanics. Now Mercury moves at the greatest velocity of any of the planets and it is precisely Mercury that possesses a small anomaly not yet explained. The new mechanics accounts for a part of this, as Lorentz has shown, and nowhere else produces a sensible modification in the motion of the planets.* After these facts have been presented, Poincaré concludes by observing that the Newtonian mechanics will remain forever the mechanics at velocities which are small with respect to the velocity of light, and thus will continue to preserve its fundamental importance.

G. D. BIRKHOFF.

THE THEORY OF ELECTRONS.

The Theory of Electrons and its Applications to the Phenomena of Light and Radiant Heat. By H. A. LORENTZ. Leipzig, B. G. Teubner, 1909. 332 pp.

Ueber Elektronen. Von W. WIEN. Zweite Auflage. Leipzig, B. G. Teubner, 1909. 39 pp.

THE physical hypotheses which are adopted for the formulation and discussion of the correlation of a certain restricted group of physical phenomena, and in particular for the development of their mathematical theory, depend to a large degree upon those phenomena themselves and are in no small measure independent of other groups of related phenomena and of the point of view adopted relative to physics as a whole. Thus in dealing with most problems in heat, the old idea of heat as a sort of massless substance, caloric, still serves as the simplest

* Newcomb and Seeliger have shown that the anomaly in the motion of Mercury and of the other inner planets can be explained, for the most part, by the attraction of matter, diffusely distributed about the sun, of plausible mass, and of such distribution as to give rise to the known phenomenon of the zodiacal light.

hypothesis, and is and should be used despite the facts that for some phenomena in heat the kinetic theory is necessary and that for the proper point of view relative to physics as a whole heat is regarded as a mode of motion. A similar, though more highly diversified, state of affairs exists in regard to electricity. There is still ample field for the application of the old hypothesis of one or two electric fluids, there is still occasion to avail oneself of the hypothesis of action at a distance as well as of that of action through a medium, and furthermore there are now coming more and more into the foreground some phenomena which are best treated by adopting the hypothesis of atomic electricity — at any rate as far as negative electricity is concerned. It is with this latest hypothesis, this atomistic view of electricity that Lorentz's Theory of Electrons deals.

The application of the atomic hypothesis to electricity is by no means of very recent date. It was foreshadowed already in Faraday's work on electrochemical equivalents; it was stated with clearness and precision by Helmholtz in his Faraday Lecture nearly thirty years ago; it has for a long time been the subject of numerous theoretical and experimental researches on the part of many physicists of whom J. J. Thomson and Lorentz are perhaps the most eminent. In the last fifteen years the development and application of the atomic view has reached such proportions as to constitute of itself a vast science. Some years ago the publication by H. Abraham and P. Langevin of their large collection of memoirs by various authors on the general subject of ions, electrons, and corpuscules * did much to render the literature easily accessible; but, as a connected, theoretical, and detailed account of electrons and their applications has for some time been sadly needed, it is particularly fortunate that we have now just such a work from the pen of one who is himself the founder and builder of so great a part of the whole theory. The lectures which formed the foundation of the book under review were delivered in this country at Columbia University in 1906. The years between the delivery of the lectures and their publication saw many advances in the atomistic theory of electricity and of these advances those which are most germane to the point of view of the lectures have been incorporated in the volume.

* Les Quantités élémentaires d'Électricité—Ions, Électrons, Corpuscules—mémoires réunis et publiés par H. Abraham et P. Langevin, Paris, Gauthier-Villars, 1905. xvi + 1136 pp.

Chapter I deals with general principles and the theory of free electrons. The author adopts the same system of units and the same notation as in his articles in the fourth volume of the Encyclopædia; he uses vector analysis. It is interesting to notice the attitude he takes toward vector analysis — about the same attitude as he takes toward the choice of units — one of convenience. Nearly two pages are required to construct his system of vector analysis. The next thing to do is to set up the equations of the free ether. Then, after mentioning the possibility of deriving the equations for ponderable bodies (at rest) by the introduction of certain constants, the author settles down to showing the necessity of the electronic or atomistic point of view and to setting up the desired equations for the electrons. The integration of the equations by means of the scalar and vector retarded potentials is given. With this groundwork everything is ready for a rapid development of such vital points in electromagnetic theory as the Poynting vector and the flow of energy, stresses in the medium, the pressure of radiation, electromagnetic momentum, (approximately) uniform motion of a charge, electromagnetic mass, the vibration of a charge and the radiation of energy, the electromagnetic equations referred to moving axes, and the kinetic theories of Drude. The arrangement of the text is noteworthy. The details of mathematical analysis are relegated to notes at the end of the book (where they occupy 100 pages) and thus the main text is kept free for the presentation and discussion of the results. The same method is pursued in the later chapters. There is nothing dogmatic about the handling of the material, the author discusses his problems frankly and with every attempt to explain the why as well as the how.

It is but natural that the first chapter should be rather plain sailing and that its chief interest should lie in the artistic presentation of the necessary introductory material. With the second chapter, on emission and absorption of heat, we come to the consideration of some of the most troublesome questions of modern physics, lying in the widely different fields of kinetic theory, thermodynamics, and electromagnetism. The author treats Kirchhoff's law, the Boltzmann-Stefan law, and Wien's displacement law. He then turns his attention toward the further determination of the law of distribution of energy in the spectrum of a black body. Here three different theories are presented and discussed — that of Planck, which leads to

the most complete determination of the function but is not treated in detail because it is hardly germane to the method of this book, that of the author himself, which is a beautiful combination of kinetic and electric theories but is valuable only for large wave lengths, and that of Jeans, which is more dynamic in nature but does not admit a strict state of equilibrium. The discussion and comparison of these theories is clear and illuminating. Despite the excellencies of the theories of Jeans and of the author, our own inclinations are strongly toward that of Planck; its foundation upon the rather vague mechanism of the resonator may be hardly electromagnetic, its use of the theory of probability may be scarcely canonical, its implied quasi-atomic structure of energy may be distinctly annoying, but somehow it does get very close to the facts and in time its difficulties may evaporate.*

Chapter III on the theory of the Zeeman effect is a very pretty piece of work. After discussing the simplest case of the Zeeman effect the question of series of spectral lines is treated. There then follows an examination of the general properties of a vibrating system of electrons and of the special properties of certain simple systems. The difficulties of a complete explanation of the phenomena involved are thoroughly presented. The author is bent on stating the importance of the problem and the need of its solution much more than on over-emphasizing the value of work already accomplished; he even concludes that "after all, we are rather at a loss as to the explanation of the complicated forms of the Zeeman effect." There can be little doubt, however, that he is on the right track and has contributed largely to the ultimate resolution of the difficulties of the analysis of the spectrum. The very diversified spectra of different elements, the astonishing simplicity of some and the bewildering complexity of others, will occupy the attention and perhaps thwart the endeavors of mathematical physicists for a long time to come.

Intimately connected with the foregoing and an immediate extension of it is the work of Chapter IV on the propagation of light in a body composed of molecules and the theory of the

* It may be remarked that Larmor has recently given an exposition of Planck's theory which apparently removes the difficulty as to the structure of energy. Another most remarkable fact is that, although Planck by his method found a value for e (4.7×10^{-10}) about fifty per cent. larger than that which had long been accepted, some recent investigations seem to corroborate his theoretical value. A showing like that means a great deal.

inverse Zeeman effect. The author passes from his equations for electrons to those for ponderable bodies by a system of averages executed over large numbers of electrons and he is able to give the theory of dispersion of light with the particular hypothesis that the electrons, which are now so well known to be fundamental, are those small sensitive particles which occur with no specializing hypothesis in the theories of Sellmeyer and others. Although the author makes no mention of Gibbs, it may be well to recall that Gibbs, in a series of papers, remarkable for generality rather than specialization of hypothesis and culminating in his deduction of the general equations of monochromatic light in media of every degree of transparency, constructed an electromagnetic theory of optics on a foundation of averages. In a problem like this it is decidedly advantageous to distinguish those results which may be obtained by general consideration from those which follow from special hypotheses. The author's work with electrons leads to a number of formulas which may be compared with experimental evidence to show their accuracy and to emphasize the value and essential truth of the physical model which he uses. After touching upon the relation of Faraday's rotation of the plane of polarization to the Zeeman effect and upon some of Voigt's results in magneto-optics, the author again closes with his characteristically modest warning that the theories he has set forth can hardly be regarded as more than a start toward the final solution of the problems. A perusal of the chapter will, however, convince the reader that he is reading some very real up-to-date vital physics.

The last chapter is devoted to optical phenomena in moving bodies. Here again, as constantly throughout the volume, we are in the presence of theories which are essentially due to the initiative of Lorentz. Here again there is no haste to assume any particular hypothesis without due consideration; the points of view of Stokes and Fresnel are examined before passing to the strict theory of electrons. The experiments of Michelson and Morley and of Rayleigh and Brace are discussed. The significance of the transformations of what Poincaré has called the Lorentz group is made clear. Some remarks relative to the rigid electron of Abraham and the electron of constant volume of Bucherer and Langevin are inserted. Finally the work closes with a brief reference to the principle of relativity as formulated by Einstein. At present the principle of relativity,

owing to pioneer work of Einstein and Minkowski and the enthusiastic cultivation of their followers, is receiving a great deal of attention. Lorentz speaks somewhat regretfully of the fact that he cannot give it greater space in his book. But we must remember that the methods employed by Einstein, Minkowski, and others in developing this principle are Maxwellian rather than Lorentzian, that they are not essentially concerned with electrons, and that consequently they are not really a proper part of the present work.

In fact, what some students of relativity would do with electrons, especially with vibrating electrons emitting trains of waves, is hard to imagine; for there are not a few who seem intent on abolishing the ether and on disregarding the phenomena which were the prime cause for its introduction. We may all come to this — one never can tell. If Lorentz's ether-electron model of the universe is so highly valuable as a basis for some domains of physics, there may well be other nearby domains for which the etherless kinematic space of some of the relativists is equally useful. And it will not do to overlook J. J. Thomson's Faraday-tube model! To the mathematical purist it may seem frivolous to take such a broad, smiling attitude toward the coexistence of such different models in a single scientific field; that is because the pure mathematician is an esthesiogenist rather than a scientist. So careless is the real scientist in the presence of divers models in the same field that we may safely venture the guess that even among specialists on heat, thermodynamics, and kinetic theory there are few who from the kinetic point of view can readily establish the equations for the flow of caloric in a marble column; they have more important and interesting things to do.

To pass from Lorentz's work to the little essay of Wien is but a step. It was in 1906, we believe, that Wien delivered an address before the Versammlung deutscher Naturforscher und Aertze on electrons. The address was printed, and has now appeared in a second edition to which some material of recent date has been added. As the pamphlet is itself a very succinct review of a wide field, it would be futile to review it here further than to mention its popular and yet scientific nature which must recommend it to all who are interested in what is going on in live physics and who have not yet acquired Wien's broad view over the field of electron theory. For those

really unfamiliar with the subject it would be well to read Wien before Lorentz and then again afterward. It is an excursion that is worth while for anyone. Come, drop an epsilon just for once and pick up an electron; it is a deal larger, even if its radius is only 10^{-13} cm.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *August*, 1910.

SHORTER NOTICES.

Vorlesungen über Differentialgeometrie. By R. v. LILIENTHAL. Erster Band: *Kurventheorie.* Leipzig, B. G. Teubner, 1908. 8vo. vi + 368 pp. 12 Marks.

IT is undoubtedly one of the achievements of the mathematical development of recent years to have shown that mathematical rigor is not the same as pedantry; that a theory may be presented in an irreproachable manner and still be attractive and interesting. There can be little doubt from a pedagogic point of view that if, in a treatise intended for students, a choice must be made between a comprehensive but tedious, and a less comprehensive but interesting treatment, the latter should be given preference, provided of course that its limitations be properly indicated. Both of these general principles seem to have been ignored completely in the construction of the book under review.

The author wishes to be rigorous and general. For this purpose he excludes all considerations involving infinitesimals. One might begin to quarrel with him on this score, since a properly formulated notion of infinitesimals is not altogether unknown. He then confines his attention exclusively to the case in which all of the functions which occur are analytic, a restriction of generality which seems extraordinary from the author's point of view, since all of his developments require only the existence of a finite number of derivatives. But even then, also for the sake of rigor, he excludes (as a matter of course, he says), the consideration of questions of order of contact, although all of these questions are easily treated by power series methods and would fall most naturally into the theory as developed by him.

It is true, historically, that differential geometry has up to the present time been occupied primarily with metric questions. We should doubtless also exercise a certain amount of patience with those who are accustomed to think of differential geometry as exclusively metric, and who refuse to bother with projective considerations in their own work. Still, in a comprehensive treatise on differential geometry, either the projective theory should be included, or else the exclusion should be explicitly indicated by the introduction of some qualifying adjective into the title; we should suggest *metric* differential geometry. The author does, however, mention briefly one phase of projective differential geometry by discussing the curves which belong to a linear complex. The following amusing remark is of interest, however, in showing how far removed he is from the modern spirit of geometry, in which the change from one space element to another has turned out to be so exceedingly fruitful. He says: "Die Plücker'sche Theorie ist entweder ein Beispiel dafür, dass rein formale Überlegungen zu wichtigen geometrischen Begriffsbildungen führen können, oder sie ist, erst nachdem sich ihr Urheber auf anderem Wege von der Bedeutung des Komplexbegriffes überzeugt hat, unter dem Vorbilde der gewöhnlichen Theorie der Ebene und der Flächen zweiten Grades aufgestellt." He obviously means, as the context shows, that the importance of the notion of a complex consists entirely in its applicability to kinematics.

Another omission which, to the reviewer, seems a serious one, is that no mention is made of intrinsic geometry as developed principally by Cesàro. The general theorems which are connected with this point of view, and the added strength which it gives to the investigator, are so valuable that no treatment of the theory of curves which goes beyond the elements can be considered adequate which does not at least give some account of this theory.

In spite of these criticisms, there are a number of things in the book which deserve high commendation. The treatment of one-parameter families of plane curves is very careful, and involves some new points of view, although the unusual distinction between envelope (Einhüllende) and curve of contact (Berührende) is of doubtful value.

The ordinary treatment of the differential geometry of plane curves is essentially a geometric interpretation of the first and second derivatives by means of the tangent and the circle of

curvature. A geometric interpretation of the third derivative is due to Abel Transon (1841) who introduced the notions abberrancy of a curve, and axis of aberrancy. This notion has been almost completely lost, and the author is to be commended for reviving it. Transon's term, however, was "axis of deviation," which just reverses the historical statement as given by the author in regard to these two names.

The book is carefully printed and none of the misprints noticed by the reviewer can give rise to any difficulty.

E. J. WILCZYNSKI.

Elliptische Funktionen. Von Professor Dr. KARL BOEHM. Erster Teil. Göschen (Sammlung Schubert XXX). Leipzig, 1909. xii + 354 pp. 8.60 Marks.

A TREATISE on any of the functions of analysis, the properties of which are well known, must rely for its usefulness upon the mode and style of presentation. The historical development may be followed, or the functions may be introduced through later discovered properties. There can be no doubt that the former is the natural and more easily comprehended introduction, especially to the higher functions. Professor Boehm has elected the latter course, in this first volume, with the understanding, however, that the reader may commence with the second volume which starts out with elliptic integrals and the inversion problem.

The present volume is occupied with the various infinite series which represent simply and doubly periodic functions, with related series and products, and with their mutual interdependence.

The beginner will probably do well to take the author's suggestion and commence with the second volume. Students who have had a good course in the calculus can easily appreciate the inversion problem and its close proximity to so-called applications, but would most likely become discouraged and cry cui bono if requested to assimilate the contents of this volume in order to become acquainted with elliptic functions. This is true even if the shorter course were followed which the author has carefully planned and indicated by footnotes at the proper places through the volume.

It must be said, however, that the volume contains all the necessary preparation for an understanding of the series repre-

senting the elliptic functions. Indeed we do not meet with these series until we have read nearly one-third the volume, or through the first four chapters. These introductory chapters are occupied with infinite series and infinite products in general and with the particular series and products which represent simply periodic functions — the latter for the purpose of analogy. The author takes advantage of these early pages to introduce ideas and a terminology peculiar to the theory of functions in general and to periodic functions in particular. Thus the reader learns about zeros and poles, periodicity and homogeneity, even and odd functions, polarized functions (after Méray), holomorphic and meromorphic functions, etc.

We begin the study of the series leading to the elliptic functions at the fifth chapter. The author avoids the Mittag-Leffler theorem (it is mentioned with a reference to the *Acta Mathematica* on page 34) and considers de novo the convergence of the series

$$\sum_{n=-\infty}^{n=+\infty} \sum_{m=-\infty}^{m=+\infty} [n + m\pi + n\theta]^{-h}.$$

This is also done for the analogous simply infinite series in the introductory chapters. The work seems unnecessarily long and tedious in consequence, but a deeper insight into the nature of the convergence is perhaps thereby gained. For example, in section 46 the above series for $h = 2$ is considered between the *finite* limits $-n_1, +n_1; -m_1, +m_1$. The limit toward which the series converges for infinitely increasing values of m_1 and n_1 is shown to depend upon the limit of the ratio $m_1 : n_1$ and is finite only when the latter limit is finite. It is doubtful, however, if the knowledge gained is worth the effort to master the longer, if more elementary, processes entailed as well as the sacrifice of a definite statement and proof of the Mittag-Leffler theorem as a point of departure. At the end of the chapter, which is forty pages in length, we are in possession of the conditions for convergency of the basal series for $h = 1, 2, 3$, together with some of the properties of the functions to which these series lead; i. e., ζu, pu, $p'u$.

Chapter six is given up to the consideration of the sigma function. The same general method is followed, beginning with the consideration of the convergency of the proper doubly-infinite product and ending with the development in trigonometric functions.

After an excellent chapter on congruent complex numbers with respect to a double modulus, we come in chapter eight to the first explicit definition of an elliptic function, viz., an elliptic function is a doubly periodic function which is meromorphic over the finite region. Then follow the Liouville theorems concerning the sum of the residues, the equality in number of zeros and poles, and the congruency of the sum of zeros to the sum of the poles in a period parallelogram.

We have now completed the groundwork of the theory and it remains to fill in the details. Some points in the arrangement of this detail may be noted. The development of the primary Weierstrassian functions in power series is put off to the eleventh chapter after a long discussion (chapter ten) devoted to the less readily comprehended Hermitian elliptic functions of the second and third kinds. Again the Jacobi theta functions, the sigmas with index, and the Gudermanian *snu*, *cnu*, *dnu*, are not introduced until the twelfth and last chapter.

One might naturally hesitate to adopt this arrangement in presenting the subject to students for the first time, even if the above series were elected as the point of departure. A more or less ready familiarity with the primary functions and the modes of reducing them to numerical computation would seem advisable before proceeding to the more complex functions which can be expressed in terms of them.

On the other hand, it must be said that there is a certain gain in elegance when all the elliptic functions of first, second, and third kinds are introduced by means of their formal developments in infinite series and the property of periodicity together with a proof that they can all be expressed in terms of a relatively small number of primary functions. The detailed discussion of these primary functions would then follow and complete the course. That Professor Boehm has, in the main, adopted this latter arrangement of material makes his book appeal more to the advanced reader than to the beginner.

The book is consistent throughout in adhering to the main line of thought above sketched and thus avoids confusing the reader by presenting a number of points of departure with their consequent developments. While one may not agree with the author as to arrangement of material, one must admire the style and general effect of the book. It is interesting and valuable to have consistent and parallel developments of the simply and doubly periodic functions brought together in a convenient form.

L. WAYLAND DOWLING.

Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung. Von FRIEDRICH DINGELDEY. Erster Teil: *Differentialrechnung.* Leipzig and Berlin, Teubner, 1910. (Sammlung Teubner, Band XXXII, 1). vi+202 pp.

THIS volume, the first of two covering the field of the calculus, contains a collection of 580 exercises illustrating the application of the differential calculus to the problems of geometry, physics, and engineering. The author is warranted in making the assumption that in the collections made heretofore the problems in the three fields have been too much isolated; and, working with the conviction that they should be united in one volume, he has arranged in the present book a most excellent combination of such geometric problems as might interest the physicist and engineer, together with physical and technical applications of the differential calculus which ought to appeal to the students of pure mathematics.

The book contains not only the exercises to be solved, but also all necessary explanations, indicated steps in the solutions of some of the problems, and answers to all. At the beginning of each chapter is given a statement of the principles of the calculus to be used, which is more than a collection of rules to be applied arbitrarily, and which falls short of a treatise on the theory. The specific purpose of the book is to supplement, not to furnish, the theory of the calculus. Numerous references to mathematical journals and published texts should inspire a further investigation on the part of the student into the problems proposed. Since it is intended that the book be used in connection with a first course in the calculus presented by lectures, there is necessarily quite a list of drill exercises included in the twenty chapters, covering the usual subjects discussed in a thorough first course in the differential calculus. However, the problems, for the most part, are of a type above furnishing mere drill in applying principles and possess mathematical interest of their own, while many may be called classics in the study of the calculus. These are well suited for, and could be used to great advantage in connection with a second course in that subject. A systematic study of the various steps in the solutions of these exercises would be of the greatest benefit to the student, and would exhibit clearly the many fields of mathematics entered in the study of the calculus. A glance at the index reveals such terms as absolute convergence, Bernoulli's numbers, Wronskian and functional determinants,

Hessian curves, least squares, line coordinates, potential, and the like, all of which when used are accompanied by explanations of their nature sufficient for the purpose of the problems.

While the author in the preface would give the impression that the fields of geometry, physics, and engineering are equally represented in the collection, yet even a casual glance reveals that the geometric side of the calculus is emphasized more than the other two, and an abundance of interesting geometric properties of curves and surfaces is brought out. Problems involving numerical calculations and the use of tables are brought in frequently, a procedure calling for definite results, and well worth while.

Instructors of the calculus will find in the collection much material to supplement any course given where new problems are welcome; while authors of text-books on the subject might easily receive many valuable suggestions from a perusal of its pages.

The volume is in every way typographically, both in print and figures, up to the high standard of the Sammlung Teubner; in its specific field it fills a practical need, and because of the excellence of the first part the volume on the integral calculus is eagerly awaited.

ERNEST W. PONZER.

Differential and Integral Calculus. By PROFESSOR DANIEL A. MURRAY. New York, Longmans, Green and Company, 1908. xviii + 491 pp.

THE aim of the author of this text-book has been (to quote from the preface) "to describe and emphasize the fundamental principles of the subject in such a way that, as much as may reasonably be expected, they may be clearly understood, firmly grasped, and intelligently applied by young students"; and again, "the aim has been to write a book that will be found helpful by those who begin the study of calculus without the guidance and aid of a teacher." This is by no means a simple undertaking in view of the inherent and essential difficulties of the subject. The notion of a limit is fundamental; and while students readily acquire more or less vague ideas on this subject, it seems to be difficult for most of them to get a clear and accurate conception of it. This difficulty must be squarely faced; fundamental definitions and principles must be set forth in language that is accurate and therefore necessarily technical.

Any attempt to state, restate, or explain these principles in language that is less technical, and therefore inaccurate or even meaningless, is not conducive to real clearness. The beginner in calculus is not helped by having the inherent difficulties of the fundamental concepts explained away. And into this sort of error, the author of the book under consideration would seem, in the judgment of the reviewer, to have fallen to some extent.

A few specific cases may be cited in support of this opinion. In the very beginning, the limit idea cannot be avoided in defining the derivative, for the reason that the fraction $\Delta y/\Delta x$ is *meaningless* when $\Delta x = 0$. When one writes, for example,

$$y = x^2, \quad \Delta y = 2x\Delta x + \Delta x^2, \quad \frac{\Delta y}{\Delta x} = 2x + \Delta x,$$

the third equation is true *except when* $\Delta x = 0$. The author seems to make no distinction between the two members of this equation. Thus, on page 6, we read, "$\Delta y/\Delta x$ will reach the value 4 when Δx decreases to zero," but $\Delta y/\Delta x$ has not been defined to mean anything other than Δy divided by Δx. On page 9 one finds, "one can confidently and accurately state what these ratios [$\Delta s_1/\Delta t_1$ and $\Delta y_1/\Delta x_1$] *will become* [the italics are from the book] when Δt_1 but Δx_1 actually reach zero." This latter case is hardly excused by a note immediately following which says, "Moreover, it should be carefully noted that *at the final stages* in the solutions of the problems in Arts. 3 and 4, $\Delta s_1/\Delta t_1$ is *not* regarded as a fraction composed of *two* quantities, Δs_1 and Δt_1, but as a *single* quantity, namely the *speed* after t_1 seconds." The symbol was certainly introduced into the problem as a fraction. Just when does it cease to be a fraction?

On page 29 we read, "It is evident that the reciprocal of an infinitesimal approaches a number which is greater than any number that can be named, namely, an infinite number. Accordingly, an infinite number may be defined as the reciprocal of an infinitesmal." Does this mean anything? It may be allowable to use the notation $x = \infty$ *after one has explained exactly what such notation is to mean;* but it is doubtful whether there is ever justification for writing (as the author does, page 185) $F(a) = \infty$, or for speaking of "the equation, $\phi'(x) = \infty$" (page 117). An excellent definition of an asymptote as the limiting position of a tangent is given on page 200; but later,

on page 202, we read that a certain line "is tangent to the curve at the infinitely distant point."

Other examples of rather loose reasoning are noticeable. On page 47, having proven that

$$\frac{d}{dx}[f(x)+c]=\frac{d}{dx}[f(x)],$$

the author says, "It follows from (1) [the equation just cited] that the derivative of a constant is zero"; but the distributive law has not yet been proven. In a note at the bottom of page 117 we read, "It follows . . . that if $\phi(a)$ is a maximum (or minimum) value of $\phi(x)$, then . . . $\sqrt{\phi(a)}$ is a maximum (or minimum) value of $\sqrt{\phi(x)}$," which is obviously not true for $\phi(x)=x^2$. The argument on page 275 holds only for a continuously increasing function, which restriction is not stated in the text. The statements (page 200) that "an ellipse cannot have an asymptote" and "the parabola $y^2=4px$ has no asymptote" are flat contradictions of the statement (page 203) that "a curve whose equation is of the nth degree has n asymptotes, real or imaginary." But the reader is still more likely to be confused when he finds (page 163) an unqualified "proof" of a property of the evolute followed by a note saying, "Property (b) should not be applied thoughtlessly; for in certain circumstances, for either the curve or its evolute, the property does not hold." An instance is then cited where the property "does not hold"; but no explanation is given of this paradox of having proven something which is not true. The note is concluded by the phrase, "which is obviously absurd"; and this is certainly true in a sense probably not intended.

One's estimate of the text-book will depend entirely on whether he regards such errors as are cited above as serious and fundamental or merely trivial — if, indeed, he regards them as errors at all. The book certainly has many commendable features.

There are numerous references throughout, both to other texts of the same grade and to books where the subjects are treated more fully and rigorously. These will undoubtedly prove helpful to student and teacher alike.

Taylor's theorem is derived by extending the mean value theorem; thus giving emphasis to the remainder from the beginning, which is most desirable. Newton's application of

the mean value theorem to the finding of approximate roots of equations is given, and this feature should prove to be of practical value, especially when applied to transcendental equations. The examples given are unfortunately largely rational algebraic equations, for which Horner's method is probably simpler.

The chapters relating to space geometry and kinematics are interesting and attractive. A nice distinction is made between speed and velocity, leading to the notion of normal as well as tangential components of acceleration.

The part devoted to integral calculus includes a chapter on center of mass and moment of inertia. This is followed by a chapter containing a brief treatment of ordinary differential equations of the first and second orders.

An appendix contains notes on hyperbolic functions, the intrinsic equations of a curve (in the plane), and the length of a curve in space; also a number of exercises and problems, a brief table of integrals, a useful collection of figures of curves, and a very complete index.

The mechanical features of the book are attractive, and it is conspicuously free from typographical errors.

WALTER B. CARVER.

An Elementary Treatment of the Theory of Spinning Tops and Gyroscopic Motion. By HAROLD CRABTREE. London, Longmans, Green and Company, 1909. 8vo, xii + 140 pp.

"THE object of this book is to bring within the range of the abler Mathematicians at our Public Schools and of First Year undergraduates at the Universities [in England], a subject which has hitherto been considered too difficult for any but the more advanced students in Mathematics, while even they have in many cases failed to derive more pleasure from the study of spinning tops than is contained in submitting the problem to the action of a complicated piece of Mathematical machinery which automatically, though unintelligently, turns out the correct result."

This extract from the preface shows well the character of the greater part of the book. At the present time, the interest of the subject, in view of recent developments, renders such an exposition of value to an engineer. But it may be read with profit by those who wish to obtain some idea of the grade reached by

the English student in his freshman year. It should be stated that the elements of statics and dynamics developed from the Newtonian laws of motion are considered as essential at an early stage as trigonometry, and they are usually commenced before the calculus. They are, however, developed from this point of view with the same degree of rigor which we usually adopt for the earlier parts of pure mathematics. The student thus imbibes the idea that physical phenomena are not isolated facts but rather the consequences of a few general laws.

Although the author starts from the laws of motion, apparently for the sake of logical completeness, it is implicitly assumed that the student is familiar with these laws and that he has had some practice in their applications. From this stage the fundamental property of the top is rapidly obtained: — that the main effect of a couple acting on the axis of the top is a motion of the axis in a plane perpendicular to the plane of the couple, provided the axis is free to move in any direction. An excellent collection of illustrations then follows. Besides the ordinary tops and gyroscopes, familiar to everyone, we find descriptions of Brennan's monorail system, Schlick's method of steadying ships at sea, the steering of torpedoes, the manner in which a cat always manages to land on its feet, the motion of a celt, the game of diabolo, and so on. These descriptions are always followed by explanations sufficient to show the manner in which the gyroscopic effect is involved. There are also numerous other examples taken from familiar experiences with bicycles, engine wheels, hoops, etc. In a few chapters at the end we find the usual mathematical developments.

Some obscurities likely to convey a wrong impression call for mention. On page 36, the text would lead one to infer that the orthogonal projection of an area was the same as the area seen from a point at a finite distance. The explanation on pages 1, 2 of the quasi-rigidity of a rapidly revolving chain or paper disc is unsatisfactory; surely it involves only the ordinary effect of inertia. On page 113, the simple statement that angular momentum is a vector quantity is hardly sufficient for the deduction of the rates of change with respect to moving axes: the author has mentioned no general theorem with respect to vectors.

These are, however, minor points. Mr. Crabtree's effort is decidedly successful and can be recommended as a useful addition to the working library of a department of mathematics,

physics or engineering. The volume is well and fully illustrated with plates and diagrams. An attractive and luminous style has aided the author very considerably in his desire to make the subject clear; in this connection his dedication is not unworthy of notice.

E. W. BROWN.

Dynamics of a Particle and of Rigid Bodies. By S. L. LONEY. Cambridge, University Press, 1909. 8vo. 374 pp. $4.00.

PROFESSOR LONEY'S mathematical texts are so well known that it is sufficient to say that the high standards of the earlier books have been maintained in the present volume. The text is divided into two equal parts (184 pages each), the first of which is devoted to dynamics of a particle. In the dynamics of a rigid body the author has confined himself chiefly to motion in two dimensions. There is a short chapter on motion in three dimensions, and Lagrange's equations in generalized coordinates are developed for both finite forces and blows. The reader is supposed to have a working knowledge of the calculus, but the differential equations which occur are solved in the text, and an appendix contains a summary of the methods of solution.

It is assumed that the student has previously read a course in elementary dynamics and the present book is mainly a treatment of more difficult problems than occur in a first course. There is a large collection of very interesting examples which furnish excellent material for the cultivation of ability in the solution of problems.

The text has been made as brief as possible and in some places the prerequisite knowledge is greater than any American student is likely to have. For example on page 76 we find that the reader is supposed to be familiar with the (p, r) equations of the conics, where r denotes the distance from the origin O to a point P on the curve, and p is the distance from O to the tangent at P. A recent notice of an American book in an English publication* contains the remark: "In looking through the American text-book a British teacher is almost always puzzled as to what previous equipment the reader is supposed to possess." This statement implies a criticism of recent tendencies in our teaching of mathematics which may or may not be important, depending on the course of study pur-

* *Mathematical Gazette*, March, 1910, p. 215.

sued. If the prospective student of higher mathematics has taken only the ordinary college courses in trigonometry and analytic geometry, he is not provided with machinery to facilitate the work in hand. It is not likely that he can make free use of the hyperbolic functions, and he may have difficulty with oblique axes. The equation of an ellipse referred to a pair of conjugate diameters will mean very little to him if he has never heard of conjugate diameters. To such a student this volume would be very valuable as a source of suggestion for further study in the subjects which ordinarily precede the work in mechanics.

W. R. LONGLEY.

NOTES.

THE annual meeting of the London mathematical society was held on November 10. Reports for the preceding year were presented, and the following officers elected: president, H. F. BAKER; secretaries, A. E. H. LOVE and J. H. GRACE; treasurer, Sir J. LARMOR; four vice-presidents, and nine other members of the council. The following papers were read at this meeting: By W. D. NIVEN (presidential address), "The relation of mathematics to experimental science"; by G. H. HARDY, "Properties of logarithmico-exponential series"; by G. T. BENNETT, "The double six of lines"; by W. H. YOUNG, "On semi-integrals and oscillating successions of functions," and (with Mrs. YOUNG), "On the existence of a differential coefficient"; by F. TAVANI, "The analytical extension of Riemann's zeta-function"; by T. W. CHAUNDY, "The geometrical representation of non real points in space of two and of three dimensions"; by J. E. LITTLEWOOD, "The extension of Tauber's theorem"; by W. H. YOUNG, "A note on the property of being a differential coefficient" by F. B. PIDDUCK, "The stability of rotating shafts"; by J. W. CHAPMAN, "On non-integral orders of summability of series and integrals"; by A. A. ROBB, "Optical geometry of motion"; by A. R. FORSYTH, "Lineo-linear transformations, especially in two variables"; by W. H. YOUNG, "On the condition that a trigonometric series should have the Fourier form."

At the Basel meeting of the Swiss mathematical society Professor Rudio reported on the organization and progress of the committee to superintend the publication of Euler's works. Editors have been selected for all the 43 volumes, and a large part of the material has already been distributed among them. Complete files of the publications of the academies of Berlin and St. Petersburg have been secured, and a complete set of the books of Euler which were published separately was presented to the commission. A set of 165 packages of manuscripts, hitherto in the possession of the St. Petersburg academy, have been received and found to contain a wealth of new material. A large number of portraits have been collected, which will be reproduced in the leading volumes of each series. Teubner, of Leipzig, has been selected as publisher. Three volumes are now in the press and will appear in a few weeks. Further volumes are in preparation and will be issued during 1911. The commission has received subscriptions for 350 sets, and fr. 110,752 in cash.

The firm of Martin Schilling in Halle announces the following additions to its collection of mathematical models: Series XXX, number 8, plaster model of the surface

$$z = xy.\frac{x^2 - y^2}{x^2 + y^2},$$ for which $$\frac{\partial^2 z}{\partial x \partial y} \neq \frac{\partial^2 z}{\partial y dx}$$ at the point (0, 0, 0),

by W. Fischer, with the cooperation of Professors Schilling and Sommer, of the technical school at Danzig; series XXXVI, numbers 1–4, four models to represent affine transformations in the plane and in space, by Professor F. Klein; series XXXVII, numbers 1–4, four pasteboard models of regular reentrant polygons, by Professor F. Schilling and Dr. O. Wiessing; series XXXVIII, number 1, model to illustrate the theory of the linear line complex, by Professor F. Schilling.

The following university courses in mathematics are announced:

Oxford University (Michaelmas term, beginning October 18, 1910). All courses are two hours per week. — By Professor W. Esson: Analytic theory of plane curves; Synthetic theory of plane curves. — By Professor E. B. Elliott: Sequences and series; Elementary theory of numbers. — By Professor A. E. H. Love: Analytic statics and harmonic analysis. — By Professor H. H. Turner: Mathematical as-

tronomy. — By Dr. J. E. CAMPBELL: Differential equations. — By Mr. H. C. THOMPSON: Integral calculus. — By H. T. GERRANS: Tridimensional rigid dynamics. — By Mr. A. L. DIXON: Hydrostatics. — By Mr. A. L. PEDDER: Problems in pure mathematics. — By Mr. C. E. HASELFOOT: Theory of equations. — By Mr. P. J. KIRKBY: Projective geometry. — By Mr. C. H. SAMPSON: Plane analytic geometry. — By Mr. A. E. JOLLIFFE: Solid geometry. — By Mr. J. W. RUSSELL: Differential calculus. — By Mr. E. H. HAYES: Elementary mechanics.

UNIVERSITY OF PARIS (second semester, beginning March 1, 1911). — By Professor E. PICARD: Functional equations of analysis and of mathematical physics, two hours. — By Professor E. GOURSAT: Ordinary and partial differential equations, two hours. — By Professor P. APPELL: General laws of motion and analytical mechanics, two hours. — By Professor E. CARTAN: Analysis and mechanics, two hours. — By Professor P. ANDOYER: Program for the higher certificate in astronomy, two hours. — By Professor J. BOUSSINESQ: Dispersion, double and circular diffraction, two hours. — By Professor G. KÖNIGS: General theory of machines, two hours.

COLLEGE OF FRANCE (academic year beginning December 1, 1910). — By Professor J. HADAMARD: Quasi-periodic functions with applications to mechanics, two hours. — By Professor C. JORDAN: Algebraic integers, particularly quadratic numbers, two hours. — By Dr. BRILLOUIN: Elasticity of isotropic and non-isotropic bodies, two hours.

AT the University of Glasgow a portrait of Professor WILLIAM JACK was recently unveiled. In commemoration of his fifty years of service as teacher of mathematics, a fund of three hundred pounds has been secured, the income of which will be known as the William Jack prize. It will be awarded to the writer of the most meritorious thesis for the degree of doctor of science in mathematics at the University of Glasgow.

PROFESSORS E. W. HOBSON and Sir J. LARMOR have been appointed moderators for Part II of the Cambridge mathematical tripos in 1911.

THE second award of the Bolyai prize founded in 1905 by the Hungarian academy of sciences (see BULLETIN, volume

12, pages 43, 212) was made to Professor DAVID HILBERT, of the University of Göttingen. The jury of award consisted of Professors König and Rados, of the academy, Professor Mittag-Leffler, of Stockholm, and Professor Poincaré, of Paris.

AT the annual meeting of the Royal society of London, the Sylvester medal was awarded to Dr. H. F. BAKER, in recognition of his treatise on abelian functions, and for his edition of the works of Sylvester. The medal is awarded biennially for the encouragement of mathematical research irrespective of nationality.

SIR GEORGE DARWIN has been elected president of the Cambridge philosophical society.

DR. P. KOEBE, of the University of Göttingen, has been appointed associate professor of mathematics at the University of Leipzig.

DR. W. KUTTA has been appointed professor of mathematics at the technical school at Aachen.

DR. J. JOUGUET has been appointed professor of mathematical analysis at the national school of mines at Paris.

PROFESSOR BEPPO LEVI, of the University of Cagliari, has been transferred to the University of Parma.

DR. A. VITERBI, of the University of Pavia, has been promoted to an associate professorship of theoretical geodesy.

PROFESSOR W. S. EICHELBERGER has been appointed director of the U. S. Nautical Almanac Office.

THE examiners in mathematics of the College entrance board for 1911 are Professors R. W. PRENTICE, of Rutgers College, and H. E. HAWKES, of Columbia University, and Mr. E. B. PARSONS, of the Boys High School, Brooklyn, N. Y.

AT the University of Nevada Dr. CHARLES HASEMAN has been promoted from an associate professorship to a full professorship of mathematics and mechanics.

MR. I. W. SMITH has been promoted to an assistant professorship of mathematics at the College of Agriculture and Mechanic Arts of the University of North Dakota.

AT Marietta College Professor H. L. COAR has been made full professor of mathematics and Lee lecturer on astronomy.

AT Syracuse University Dr. F. F. DECKER has been promoted to an assistant professorship of mathematics.

MR. R. R. SHUMWAY has been appointed assistant professor of mathematics in the University of Minnesota.

AT the State University of Iowa Dr. R. P. BAKER has been promoted to an assistant professorship of mathematics.

DR. A. M. HILTEBEITEL has been appointed instructor in mathematics at the University of Pennsylvania.

MR. W. S. PEMBERTON, of the University of Missouri, has been appointed professor of mathematics at Wheaton College, Wheaton, Ill.

PROFESSOR MAURICE LÉVY, of the College of France, died in October, at the age of 72 years.

COOPER D. SCHMITT, professor of mathematics in the University of Tennessee for twenty-one years, and dean of the University, died December 9, at the age of 51 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMSON (E.). Eine zeichnerisch wohl verwertbare Konstructionsweise des allgemeinen Kegelschnittes. (Progr.) München, 1910. 8vo. 19 pp.

BARUCH (A.). Ueber die Differentialrelationen zwischen den Thetafunktionen eines Arguments. (Diss.) Halle a. S., 1910.

CARVALLO (E.). Méthode pratique pour la résolution numérique complète des équations algébriques et transcendants. 3e édition. Paris, 1910. 4to.

CHRISTOFFEL (E. B.). Gesammelte mathematische Abhandlungen. Unter Mitwirkung von A. Krazer and G. Faber herausgegeben von L. Maurer. 1ter Band, mit einer Biographie E. B. Christoffels. Leipzig, Teubner, 1910. 8vo. 16+382 pp. M. 18.00

——. Gesammelte mathematische Abhandlungen. 2ter Band. Leipzig, Teubner, 1910. 8vo. 3+343 pp. M. 16.00

CICCO (S. DE). Sulla protogenesi dei processi matematici. Napoli, 1910. L. 2.00

DARBOUX (G.). Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. 2e édition, complétée, Paris, Gauthier-Villars, 1910. 8vo. 576 pp. Fr. 18.00

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome I, volume 2, fascicule 2: Propriétés générales des corps et des variétés algébriques, par G. Landsberg, J. Hadamard, et J. Kürschák. Leipzig, Teubner, 1910. M. 3.00

FESTSCHRIFT zur Feier des 100 Geburtstages Eduard Kummers. Mit Briefen an seine Mutter und an L. Kronecker. Herausgegeben vom Vorstande der Berliner Mathematischen Gesellschaft. Mit einem Bildnis E. Kummers. (Abhandlungen zur Geschichte der mathematischen Wissenschaften, Heft XXIX.) Leipzig, Teubner, 1910. 8vo. M. 4.00

FICK (E.) Anfangsgründe der analytischen Geometrie des 4fach ausgedehnten Raumes. 2ter Teil. (Progr.) Neuberg a. D., 1910. 8vo. 31 pp.

HARDY (G. H.). Orders of infinity. The "Infinitarcalcul" of Paul Du Bois-Reymond. (Cambridge tracts in mathematics and mathematical physics. No. 12.) London, Cambridge University Press, 1910. 8vo. 62 pp. 2s. 6d.

HELM (G.). Die Grundlehren der höheren Mathematik. Zum Gebrauch bei Anwendungen und Wiederholungen zusammengestellt. Leipzig, 1910. 8vo. 15+419 pp. Cloth. M. 14.20

IN MEMORIAM Irving Stringham. Professor Stringham's services to the university. William T. Reid. Irving Stringham. Mellen W. Haskell. Memorial resolutions: Regents of the University of California. Academic council. Students of the university. Faculty Club. Harvard Club of San Francisco. Berkeley Club. Berkeley, University of California, 1910.

PFISTNER (C.). Die einem homogenen linearen Differentialgleichungssystem zugeordneten Systeme. (Diss.) Freiburg i. Br., 1910.

REID (L. W.). The elements of the theory of algebraic numbers. With an introduction by David Hilbert. New York, Macmillan, 1910. 8vo. 15+461 pp. Cloth. $3.50

ROTHE (R.). Ernst Gerland zum Gedächtniss. Clausthal, 1910.

VEBLEN (O.) and YOUNG (J. W.). Projective geometry. Volume I. Boston, Ginn, 1910. 8vo. 10+352 pp. Cloth. $4.00

YOUNG (J. W.). See VEBLEN (O.).

II. ELEMENTARY MATHEMATICS.

ANDREINI (A. L.). Tavole dei logaritmi con tre e con quattro decimali. Seconda edizone riveduta. Livorno, Giusti, 1910. 16mo. 8+58 pp.

BALTIN (R.) und MAIWALD (W.). Kurzgefasstes Lehrbuch der Mathematik für Seminare und Präparandenanstalten. Nach den Lehrplänen von 1901 bearbeitet. 3te verbesserte Auflage. Leipzig, Teubner, 1910. 8vo. 10+240 pp. Cloth. M. 2.60

BARNARD (S.) and CHILD (J. M.). Key to a new algebra. Vol. I. London. 1910. 6s. 6d,

BELL (R. J. T.). An elementary treatise on coordinate geometry of three dimensions. London, Macmillan, 1910. 8vo. 372 pp. 10s.

BENDT (F.). Grundzüge der Differential- und Integralrechnung. 4te verbesserte Auflage. Leipzig, Weber, 1910. 8vo. 16+268 pp. Cloth. M. 3.00

BETTAZZI (R.). Lezioni di calcolo integrale. Torino, 1910. L. 5.00

BRENKE (W. C.). Textbook on advanced algebra and trigonometry, with tables. New York, Century, 1910. 8vo. 7+345 pp. $2.00

BREWSTER (C. W.). See SANDERSON (F. W.).

CHILD (J. M.). See BARNARD (S.).

CRAMER (H.). Der mathematische Unterricht an den höheren Schulen im Grossherzogtum Baden. (Abhandlungen, herausgegeben von F. Klein, Band II, Heft 4.) Leipzig, Teubner, 1910. 8vo. 4+48 pp. M. 1.60

DESCHAMPS (J.). Notes sur la géométrie analytique. Applications de nos identités fondamentales. Paris, 1910.

DINTZL (E.). Der mathematische Unterricht an den Gymnasien. (Berichte über den mathematischen Unterricht in Österreich, Heft 3.) Wien, Hölder, 1910.

DOLINSKI (M.). Algebra und politische Arithmetik. 2te, vermehrte und verbesserte Auflage. Wien, 1910. M. 5.00

DÜSING (K.). Die Elemente der Differential- und Integralrechnung, in geometrischer Methode dargestellt. Ausgabe B: mit zahlreichen Beispielen aus der technischen Mechanik von E. Preger. 2te Auflage. Hannover, Jänecke, 1910. 8vo. 11+101 pp. M. 1.90

ELSON (W. H.). See SHORT (R. L.).

ERDMANN (K.). Anfangsgründe der ebenen Geometrie. 2te, umgearbeitete Auflage. Dresden, Bleyl und Kaemmerer, 1911. 8vo. 8+283 pp. Cloth. M. 4.40

FARMER (R. C. and M. M.). Four-figure logarithms on a new graphic system, dispensing with interpolations. New York, Longmans, 1910. 12mo. Limp cloth. $0.20

FERRARIS (P.). Elementi di geometria, ad uso delle scuole medie. Vol. I. Torino, Paravia, 1911. 16mo. 192 pp. L. 3.00

FRANCHIS (M. DE). Elementi d'algebra, ad uso dei licei. Vol. II: per le 2e e 3e classi liceali. Palermo, Sandron, 1910. 16mo. 154 pp. L. 2.00

GECK (E.). Der mathematische Unterricht an den höheren Schulen im Königreich Würtemberg. (Abhandlungen, herausgegeben von F. Klein, Band II, Heft 3.) Leipzig, Teubner, 1910. 8vo. 4+104 pp. M. 1.60

GRANVILLE (W. A.). See SMITH (P. F.).

GRÜNBAUM (H.). Der mathematische Unterricht an den deutschen mittleren Fachschulen der Maschinenindustrie. (Abhandlungen, herausgegeben von F. Klein, Band IV, Heft 1. Mit einem Einführungswort von P. Stäckel.) Leipzig, Teubner, 1910. 8vo. 16+100 pp. M. 2.60

KOMMERELL (V.). Raumgeometrie: Stereometrie und darstellende Geometrie. Tübingen, Laupp, 1910. 8vo. 8+196 pp. M. 2.60

LAZZERI (G.). Manuale di trigonometria sferica. Seconda edizione riveduta. Livorno, Giusti, 1911. 16mo. 4+95 pp. L. 1.00

LIETZMANN (W.). Die organisation des mathematischen Unterrichts an den höheren Knabenschulen in Preussen. (Abhandlungen, herausgegeben von F. Klein, Band I, Heft 2.). Leipzig, Teubner, 1910. 8vo. 8+204 pp. M. 5.00

LINNICH (M.). Lehr- und Übungsbuch für den Unterricht in der Arithmetik und Algebra mit einem Anhang für den Unterricht in der analytischen Geometrie. Leipzig, Freytag, 1910. M. 2.50

MAIWALD (W.). See BALTIN (R.).

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MEIGEN (F.). Lehrbuch der Trigonometrie. (Technische Lehrhefte.) 2te, vermehrte und verbesserte Auflage. Leipzig, Gebhardt, 1910. 8vo. 3+77 pp. M. 2.40

MERCER (J. W.). The calculus for beginners. New York, Putnam, 1910. 12mo. 14+440 pp. Cloth. $2.00

MÜLLER (H. H.). See REINHARDT (W.).

NIEWENGLOWSKI (B.). Deuxième année de géométrie. Premier cycle. Paris, Delagrave, 1910. 18mo. 154 pp. Fr. 2.25

REINHARDT (W.). und MÜLLER (H. H.). Lehrbuch für den mathematischen Unterricht auf Grund der neuen Lehrpläne. 2ter Teil (Nachtrag): Trigonometrie. Frankfurt a. M., Auffarth, 1910. 8vo. 4+56 pp. M. 1.00

SALAZAR E IBÁÑEZ (M.). Problemas matemáticos. Madrid, Marzo, 1910. 8vo. P. 7.00

SANDERSON (F. W.) and BREWSTER (C. W.). A geometry for schools. London, Cambridge University Press, 1910. 8vo. 346 pp. 3s.

SCHAEFFER (A.). 1400 mathematische Abiturienten-Aufgaben mit 700 Lösungen und Anleitungen zur Lösung. Zabern, Fuchs, 1910. 8vo. 16+240 pp. Cloth. M. 3.80

SCHANZ (I. A.). Lehrbuch der Mathematik zum Gebrauch in höheren Mädchenschulen. 2ter Teil, 2tes Heft: Geometrie. Berlin, Oemigke, 1910. 8vo. 192 pp. Cloth. M. 2.80

SCHMIDT (J.). Lehrbuch der Elementarmathematik. Ausgabe für Gymnasien und Realgymnasien. 1ter Band. Wien, Hölder, 1910. 8vo. 7+336 pp. Cloth. M. 2.80

——. Lehrbuch der Elementarmathematik. Ausgabe für Realschulen. 1ter Band. Wien, Hölder, 1910. 8vo. 7+387 pp. Cloth. M. 3.30

SCHNELL (H.). Der mathematische Unterricht an den höheren Schulen im Grossherzogtum Hessen. (Abhandlungen, herausgegeben von F. Klein, Band II, Heft 5.) Leipzig, Teubner, 1910. 8vo. 6+51 pp. M. 1.60

SCHULTZE (A.) and MANGUSE (W. P.). Answers to elements of algebra. New York, Macmillan, 1910. 12mo. 38 pp. Paper. $0.20

SCHUSTER (A.). Einführung in die elementare Mathematik. (Sammlung Kösel, Band 33.) München, Kösel, 1910. M. 1.00

SHORT (R. L.) and ELSON (W. H.). Secondary school mathematics, Book 1. Boston, Heath, 1910. 12mo. 182 pp. $1.00

SMITH (C.). An elementary treatise on conic sections by the methods of co-ordinate geometry. New edition, revised and enlarged. London, Macmillan, 1910. 8vo. 460 pp. 7s. 6d.

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